Relativistic reference frames for radio science experiments with a Mercury orbiter

1 Introduction

In this document we shall summarize the main issues relevant for very high accuracy orbit determination in the case of radio science experiments with a Mercury orbiter, such as for the BepiColombo mission.

BepiColombo is an European Space Agency mission to be launched in 2014, with the goal of an in-depth exploration of the planet Mercury; it has been identified as one of the most challenging long-term planetary projects. Only two NASA missions had Mercury as target in the past, the Mariner 10, which flew by three times in 1974-5 and Messenger, which carried out its flybys on January and October 2008, September 2009 before it starts its year-long orbiter phase in March 2011.

The BepiColombo mission is composed by two spacecraft to be put in orbit around Mercury. The Radio Science Experiment is one of the on board experiments, which would coordinate a gravimetry, a rotation and a relativity experiment, using a very accurate range and range rate tracking. These measurements will be performed by a full 5-way link to the Mercury orbiter; by exploiting the frequency dependence of the refraction index, the differences between the Doppler measurements (done in Ka and X band) and the delay give information on the plasma content along the radio wave path [Iess and Boscagli 2001]. In this way most of the measurements errors introduced can be reduced by about two orders of magnitude with respect to the past technologies. The accuracies that can be achieved are 10 cm in range and 3×10^{-4} cm/s in range rate.

How do we compute these observables? For example, a first approximation of the range could be given by the formula

$$r = |\mathbf{r}| = |(\mathbf{x}_{\rm sat} + \mathbf{x}_{\rm M}) - (\mathbf{x}_{\rm EM} + \mathbf{x}_{\rm E} + \mathbf{x}_{\rm ant})| \ , \tag{1}$$

which models a very simple geometrical situation (Figure 1). The vector $\mathbf{x}_{\rm sat}$ is the mercurycentric position of the orbiter, the vector $\mathbf{x}_{\rm M}$ is the position of the center of mass of Mercury (M) in a reference system with origin at the Solar System Barycenter (SSB), the vector $\mathbf{x}_{\rm EM}$ is the position of the Earth-Moon center of mass in the same reference system, $\mathbf{x}_{\rm E}$ is the vector from the Earth-Moon Barycenter (EMB) to the center of mass of the Earth (E), the vector $\mathbf{x}_{\mathrm{ant}}$ is the position of the reference point of the ground antenna with respect to the center of mass of the Earth.

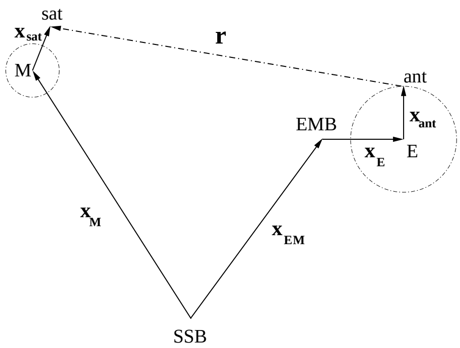

Figure 1: Geometric sketch of the vectors involved in the computation of the range. SSB is the Solar System Barycenter, M is the center of Mercury, EMB is the Earth-Moon Barycenter, E is the center of the Earth.

Using (1) means to model the space as a flat arena (r is an Euclidean distance) and the time as an absolute parameter. This is obviously not possible because it is clear that, beyond some threshold of accuracy, space and time have to be formulated within the framework of Einstein's theory of gravity (general relativity theory, GRT). Moreover we have to take into account the different times at which the events have to be computed: the transmission of the signal at the transmit time (t_t), the signal at the Mercury orbiter at the time of bounce (t_b) and the reception of the signal at the receive time (t_r).

Formula (1) is used as a starting point to construct a correct relativistic formulation; with the word "correct" we do not mean all the possible relativistic effects, but the effects that can be measured by the experiment. This document deals with the corrections to apply to this formula to obtain a consistent relativistic model for the computations of the observables.

In Section 2 we discuss the relativistic four-dimensional reference systems used, while the transformations adopted to make the sums in (1) consistent are described in Section 4; according to [Soffel et al. 2003], with "reference system" we mean a purely mathematical construction, while a "reference frame" is a some physical realization of a reference system. Section 3 deals with the appropriate time scale to be used in order to threat the mercurycentric dynamic correctly.

The equations of motion for the planets Mercury and Earth, including all the relativistic effects (and potential violations of GRT) required to the accuracy of the BepiColombo Radio Science Experiment have already been discussed in the Chapter 6.

2 Space-Time Reference frames

The five vectors involved in formula (1) have to be computed at their own time, the epoch of different events: e.g., $\mathbf{x}_{\mathrm{ant}}$, $\mathbf{x}_{\mathrm{EM}}$ and $\mathbf{x}_{\mathrm{E}}$ are computed at both the antenna *transmit time* t_t and *receive time* t_r of the signal. $\mathbf{x}_{\mathrm{M}}$ and $\mathbf{x}_{\mathrm{sat}}$ are computed at the *bounce time* t_b (when the signal has arrived to the orbiter and is sent back, with correction for the delay of the transponder). In order to be able to perform the vector sums and differences, these vectors have to be converted to a common space-time reference system, the only possible choice being some realization of the BCRS (Barycentric Celestial Reference System). We adopt for now a realization of the BCRS that we call SSB (Solar System Barycentric) reference frame and in which the time is a re-definition of the TDB (Barycentric Dynamic Time), according to the IAU 2006 Resolution B3[1]; other possible choices, such as TCB (Barycentric Coordinate Time), only can differ by linear scaling. The TDB choice of the SSB time scale entails also the appropriate linear scaling of space-coordinates and planetary masses as described for instance in [Klioner 2008] or [Klioner et al. 2010].

The vectors $\mathbf{x}_{\mathrm{M}}$, $\mathbf{x}_{\mathrm{E}}$, and $\mathbf{x}_{\mathrm{EM}}$ are already in SSB as provided by numerical integration and external ephemerides; thus the vectors $\mathbf{x}_{\mathrm{ant}}$ and $\mathbf{x}_{\mathrm{sat}}$ have to be converted to SSB from the geocentric and mercurycentric systems, respectively. Of course the conversion of reference system implies also the conversion of the time coordinate. There are three different time coordinates to be considered. The currently published planetary ephemerides are provided in TDB. The observations are based on averages of clock and frequency measurements on the Earth surface: this defines another time coordinate called TT (Terrestrial Time). Thus for each observation the times of transmission t_t and reception t_r need to be converted from TT to TDB to find the corresponding positions of the planets, e.g., the Earth and the Moon, by combining information from the pre-computed ephemerides and the output of the numerical integration for Mercury and for the Earth-Moon barycenter. This time conversion step is necessary for the accurate processing of each set of interplanetary tracking data; the main term in the difference TT-TDB is periodic, with period 1 year and amplitude $\simeq 1.6 \times 10^{-3}$ s, while there is essentially no linear trend, as a result of a suitable definition of the TDB.

The equation of motion of a mercurycentric orbiter can be approximated, to the required level of accuracy, by a Newtonian equation provided the independent variable is the proper time of Mercury. Thus, for the BepiColombo Radio Science Experiment, it is necessary to define a new time coordinate TDM (Mercury Dynamic Time), as described in Section 3, containing terms of 1-PN order depending mostly upon the distance from the Sun and velocity

[1]See the Resolution at http://www.iau.org/administration/resolutions/ga2006/

of Mercury.

From now on, in accordance with [Klioner et al. 2010], we shall call the quantities related to the SSB frame "TDB-compatible", the quantities related to the geocentric frame "TT-compatible", the quantities related to the mercurycentric frame "TDM-compatible" and label them TB, TT and TM, respectively.

The differential equation giving the local time T as a function of the SSB time t , which we are currently assuming to be TDB, is the following:

$$\frac{dT}{dt} = 1 - \frac{1}{c^2}\left[U + \frac{v^2}{2} - L\right] , \tag{2}$$

where U is the gravitational potential (the list of contributing bodies depends upon the accuracy required: in our implementation we use Sun, Mercury to Neptune, Moon) at the planet center and v is the SSB velocity of the same planet. The constant term L is used to perform the conventional rescaling motivated by removal of secular terms, e.g., for the Earth we use L_C [Soffel et al. 2003].

3 Dynamic Mercury Time

The mercury-centric orbit of the spacecraft is coupled to the orbit of the planet, mostly through the difference between the acceleration from the Sun on the probe and the one on the planet (the Sun tidal term). This coupling is weak because the Sun tide is just 10^{-7} of the monopole acceleration from Mercury. The relativistic perturbations containing the mass of Mercury are small to the point that they are not measurable, being easily absorbed by the much larger non-gravitational perturbations, measured with finite accuracy by the on board accelerometer. Should we conclude that general relativity does not matter in the computation of the mercury-centric orbit? The answer is negative, but the main relativistic effect does not appear in the equation of motion.

There are three different time coordinates to be considered. The dynamics of the planets, as described by the Lagrangian, is the solution of differential equations having a time belonging to a space-time reference frame with origin in the SSB as independent variable. There can be different realizations of such a time coordinate: the currently published planetary ephemerides are provided in a time called TDB (Barycentric Dynamic Time). The observations are based on averages of clocks and frequency scales located on the Earth surface: this corresponds to another time coordinate called TT (Terrestrial Time). Thus for each observation the times of transmission and receiving (t_t, t_r) need to be converted from TT to TDB to find the corresponding positions of the planets, e.g., the Earth and the Moon, by combining information from the precomputed ephemerides and the output of the numerical integration for Mercury and the Earth-Moon barycenter. This time conversion step is necessary for the accurate processing of each set of interplanetary tracking data; the main term in the difference TT-TDB is periodic, with period 1 year and amplitude $\simeq 1.6 \times 10^{-3}$ s, while there is essentially no linear trend, as a result of a suitable definition of the TDB.

The equation of motion of a mercury-centric satellite can be approximated, to the required level of accuracy, by a Newtonian equation provided the independent variable is the proper time of Mercury. Thus, for the BepiColombo radioscience experiment, it is necessary to define a new time coordinate TDM (Mercury Dynamic Time) containing terms of 1-PN order depending mostly upon the distance from the Sun r_{10} and velocity v_1 of Mercury. The relationship with the TDB scale, truncated to 1-PN order (we drop the $O(c^{-4})$ terms on the right hand side, that are in principle known, but certainly not needed for our purposes), is given by a differential equation

$$\frac{dt_{\mathrm{TDM}}}{dt_{\mathrm{TDB}}} = 1 - \frac{v_1^2}{2\,c^2} - \sum_{k\neq 1} \frac{G\,m_k}{c^2\,r_{1k}} \ ,$$

which can be solved by a quadrature formula once the orbits of Mercury, the Sun and the other planets are known. Figure 2 plots the output of such a computation, showing a drift due to the non-zero average of the post-Newtonian term.

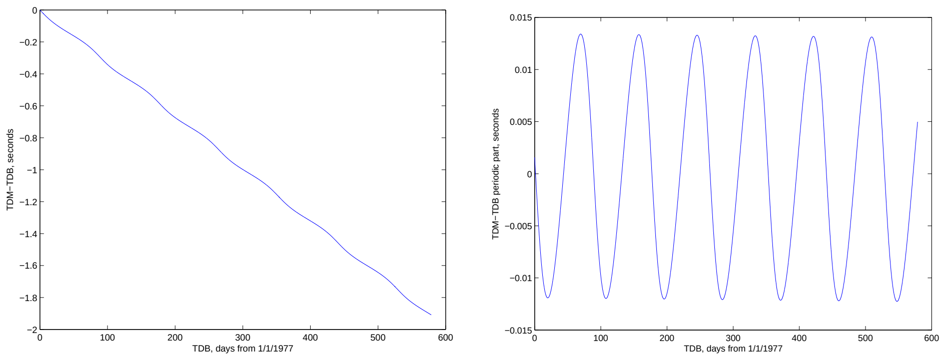

Figure 2: Left: TDM as function of TDB shows a drift due to the non-zero average of the 1-PN term. Right: the oscillatory term, with the period of Mercury's orbit, is almost an order of magnitude larger than TT-TDB.

The oscillatory term, having the one of Mercury orbit as main period, has an amplitude $\simeq 0.012$ s. In 0.01 s the spacecraft velocity can change by 3 cm/s, $\simeq 10,000$ times more than the range-rate measurement accuracy, the position by 30 m, $\simeq 300$ times the range measurement accuracy. Thus this effect has to be accurately taken into account for our experiment[2].

[2] The time scale TDB will be replaced in the planetary ephemerides by the new TCB;

4 Space-time transformations

The space-time transformations to perform involve essentially the position of the antenna and the position of the orbiter. The geocentric coordinates of the antenna should be transformed into TDB-compatible coordinates; the transformation is expressed by the formula

$$\mathbf{x}_{\mathrm{ant}}^{TB} = \mathbf{x}_{\mathrm{ant}}^{TT} \left(1 - \frac{U}{c^2} - L_C\right) - \frac{1}{2}\left(\frac{\mathbf{v}_{\mathrm{E}}^{TB} \cdot \mathbf{x}_{\mathrm{ant}}^{TT}}{c^2}\right) \mathbf{v}_{\mathrm{E}}^{TB} \ ,$$

where U is the gravitational potential at the geocenter (excluding the Earth mass), $L_C = 1.48082686741 \times 10^{-8}$ is a scaling factor given as definition, supposed to be a good approximation for removing secular terms from the transformation and $\mathbf{v}_{\mathrm{E}}^{TB}$ is the barycentric velocity of the Earth. The next formula contains the effect on the velocities of the time coordinate change, which should be consistently used together with the coordinate change:

$$\mathbf{v}_{\mathrm{ant}}^{TB} = \left[\mathbf{v}_{\mathrm{ant}}^{TT} \left(1 - \frac{U}{c^2} - L_C\right) - \frac{1}{2}\left(\frac{\mathbf{v}_{\mathrm{E}}^{TB} \cdot \mathbf{v}_{\mathrm{ant}}^{TT}}{c^2}\right) \mathbf{v}_{\mathrm{E}}^{TB}\right] \left[\frac{dT}{dt}\right] \ .$$

Note that the previous formula contains the factor dT/dt (expressed by (2)) that deals with a time transformation: T is the local time for Earth, that is TT, and t is the corresponding TDB time.

The mercurycentric coordinates of the orbiter have to be transformed into TDB-compatible coordinates through the formula

$$\mathbf{x}_{\mathrm{sat}}^{TB} = \mathbf{x}_{\mathrm{sat}}^{TM} \left(1 - \frac{U}{c^2} - L_{CM}\right) - \frac{1}{2}\left(\frac{\mathbf{v}_{\mathrm{M}}^{TB} \cdot \mathbf{x}_{\mathrm{sat}}^{TM}}{c^2}\right) \mathbf{v}_{\mathrm{M}}^{TB} \ ,$$

where U is the gravitational potential at the center of mass of Mercury (excluding the Mercury mass) and L_{CM} could be used to remove the secular term in the time transformation (thus defining a TM scale, implying a rescaling of the mass of Mercury). We believe this is not necessary: the secular drift of TDM with respect to other time scales is significant, but a simple iterative scheme is very efficient in providing the inverse time transformation. Thus we set $L_{CM} = 0$, assuming the reference frame is TDM-compatible. As for the antenna we have a formula expressing the velocity transformation that contains the derivative of time T for Mercury, that is TDM, with respect to time t, that is TDB:

$$\mathbf{v}_{\mathrm{sat}}^{TB} = \left[\mathbf{v}_{\mathrm{sat}}^{TM} \left(1 - \frac{U}{c^2} - L_{CM}\right) - \frac{1}{2}\left(\frac{\mathbf{v}_{\mathrm{M}}^{TB} \cdot \mathbf{v}_{\mathrm{sat}}^{TM}}{c^2}\right) \mathbf{v}_{\mathrm{M}}^{TB}\right] \left[\frac{dT}{dt}\right] \ .$$

when this will happen, we will use a suitably defined Mercury Coordinate Time (TCM), such that the differential equation giving the TCB to TCM conversion will be exactly the same as for TDB to TDM.

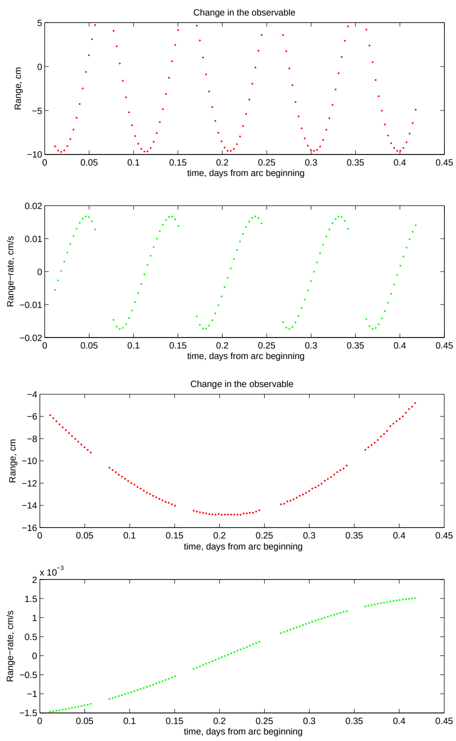

Figure 3: The difference in the observables range and range rate for one pass of Mercury above the horizon for a ground station, by using an hybrid model in which the position and velocity of the orbiter have not transformed to TDB-compatible quantities and a correct model in which all quantities are TDB-compatible. Interruptions of the signal are due to spacecraft passage behind Mercury as seen for the Earth station. Top: for an hybrid model with the satellite position and velocity not transformed to TDB-compatible. Bottom: for an hybrid model with the position and velocity of the antenna not transformed to TDB-compatible.

For these coordinate changes, in every formula we neglected the terms of the SSB acceleration of the planet center [Damour et al. 1994], because they contain beside $1/c^2$ the additional small parameter (distance from planet center)/(planet distance to the Sun), which is of the order of 10^{-4} even for a Mercury orbiter.

To assess the relevance of the relativistic corrections of this section to the accuracy of the BepiColombo Radio Science Experiment, we computed the observables range and range rate with and without these corrections. As shown in Figure 3, the differences are significant, at a signal-to-noise ratio $S/N \simeq 1$ for range, much more for range rate, with an especially strong signature from the orbital velocity of the mercurycentric orbit (with $S/N > 50$).

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