On separable systems in 2 dimensions.

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Abstract

We show how any two-dimensional natural Hamiltonian system, which is separable either at arbitrary or at fixed energy, is endowed with a Lax pair structure and can be given a bi-Hamiltonian formulation.

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1 Introduction

An active research field is devoted to geometrical methods for the investigation of integrable Hamiltonian systems: the intrinsic characterization of separability of the Hamilton-Jacobi equation with a single Killing tensor by Benenti [3] is one example. Moreover efforts are aimed at a systematization work, exploring the relationship with bi-Hamiltonian structures [4, 5, 6, 7] and the Lax pair approach [8, 17].

In all mentioned works, the metric of the configuration space is that of the kinetic energy (so that in general it is a flat space) and the attention is focused on the geometric properties of the phase-space manifold. Here we want to use the Jacobi metric on the configuration space to show how many results can be generalized or at least simplified. Geometrization of the dynamics by using the Jacobi metric [9, 10] is a standard tool to turn a natural Hamiltonian system into a geodesic flow over a conformal Riemannian manifold. Therefore, it is natural to investigate integrability of those systems by looking for invariants corresponding to Killing tensors of a conformal Riemannian geometry. In Rosquist and Pucacco [11], RP in what follows, quadratic invariants at arbitrary and fixed energy (respectively, *strongly* and *weakly* conserved phase-space functions) for 2-dimensional Hamiltonian systems were treated in a unified way. Generalizing the approach, we have discussed the existence of invariants *cubic* and *quartic in the momenta* at both fixed and arbitrary energy [14, 15].

The integrability condition for polynomial invariants involves an arbitrary analytic function $S(z)$ of complex variable. For quadratic invariants at arbitrary energy, the function $S(z)$ is a second degree polynomial with real second derivative and the integrability condition then reduces to the classical Darboux's condition for quadratic invariants at arbitrary energy [12]. In the particular case of two dimensions, every Riemannian (or even pseudo-Riemannian) metric can be expressed in conformal form (see, e.g. [16], sect. 13.1). Therefore, the results of RP are valid both in the case of Hamiltonian systems with a potential assigned in flat and curved space *and* in the case of the free flow over a given surface.

Recently, they have appeared some works covering strictly related matters. In a paper by McLenaghan and Smirnov [8], the main result consists in showing that the simplest class of two-dimensional separable systems, namely those separable in Cartesian and rotated Cartesian coordinates, admits a bi-Hamiltonian representation and their dynamics can be described in terms of the Lax pair. In a paper by Bruce et al. [17], the main result consists in presenting a classification of all separable Hamiltonian systems in two dimensional systems. Actually, the bi-Hamiltonian formulation and Lax structure of such systems have already been obtained by Rauch-Wojciechowski [1] and R.W. and Antonowicz [2]. In the present contribution, we show that, working with conformal transformations, these results are more concisely obtained along the line of RP and the bi-Hamiltonian representation and Lax description of the dynamics can be extended to *all* separable two-dimensional systems, including the fixed energy case.

2 Preliminary material

2.1 The Jacobi metric and Killing tensors

The geometrization of the dynamics in terms of the Jacobi geometry has been known for about a century and can be found in some text books [18, 9, 10]. Here we outline the main ideas to make the paper self-contained. Most Hamiltonian systems of physical interest have a geometric kinetic energy part, that is to say that the kinetic energy is a non-degenerate quadratic form in the momenta p_α

$$T = \frac{1}{2} h^{\alpha\beta} p_\alpha p_\beta, \quad \alpha, \beta = 1, \dots, n, \quad (1)$$

where $h_{\alpha\beta}$ is a function of the configuration variables q^α . The Hamiltonian itself then has the form

$$H = T + V(q) = E. \quad (2)$$

The independent variable, let us say t , is often but not always the time. For any given energy E of the system we can use the Hamiltonian

$$\mathcal{H} = H - E, \quad (3)$$

to represent the dynamics provided that we impose the constraint

$$\mathcal{H} = 0 . \quad (4)$$

For any such zero energy Hamiltonian we can reparametrize the system by introducing a new time variable $\bar{t}$ defined by the relation

$$dt = \mathcal{N}(p, q) d\bar{t} , \quad (5)$$

together with a redefined Hamiltonian

$$\bar{\mathcal{H}} = \mathcal{N}(p, q) \mathcal{H} = \mathcal{N}T + \mathcal{N}(V - E) = 0 . \quad (6)$$

The new Hamiltonian will then give the same equations of motion on the constraint surface $\bar{\mathcal{H}} = 0$. We shall use the term *lapse function* for $\mathcal{N}(p, q)$ which defines the independent variable gauge. This usage is borrowed from cosmological applications where the lapse gives the rate of physical time change relative to coordinate time. The lapse function can be taken as any non-zero function on the phase space.

The dynamics of the system can be represented in a purely geometric formulation by exploiting the reparametrization freedom. The passage to the geometric representation is accomplished by defining a new time variable t_J (*Jacobi time*) by the lapse choice

$$\mathcal{N} = \mathcal{N}_J = \frac{1}{2(E - V)} . \quad (7)$$

The corresponding Hamiltonian then has the form $\bar{\mathcal{H}} = [2(E - V)]^{-1}T - \frac{1}{2}$. Thus $\bar{\mathcal{H}}$ has a constant potential energy which can be subtracted without affecting the equations of motion. This leads to the *Jacobi Hamiltonian*, H_J , defined by

$$H_J = \frac{1}{2} \mathcal{N}_J h^{\alpha\beta} p_\alpha p_\beta = \frac{1}{2} g^{\alpha\beta} p_\alpha p_\beta , \quad (8)$$

where $g^{\alpha\beta}$ is the *Jacobi metric* of the system. It is given in covariant form by

$$g_{\alpha\beta} = 2(E - V) h_{\alpha\beta} . \quad (9)$$

The Jacobi metric is therefore conformally related to the original kinetic metric $h_{\alpha\beta}$. It follows that the dynamics of the system (2) is equivalent to geodesic motion in the Jacobi geometry

$$ds^2 = g_{\alpha\beta} dq^\alpha dq^\beta . \quad (10)$$

The simplest symmetry of the geodesic equations is known as a Killing vector field (see e.g. [19]). More generally, an *m-rank* Killing tensor is a symmetric tensor satisfying equation

$$K_{(\mu_1 \mu_2 \dots \mu_m; \mu_{m+1})} = 0 . \quad (11)$$

It gives rise to the *m-order* (in the momenta) invariant

$$I_J = K^{\mu_1 \mu_2 \dots \mu_m} p_{\mu_1} p_{\mu_2} \dots p_{\mu_m} . \quad (12)$$

Killing tensors and *m* degree polynomial invariants are the natural generalizations of Killing vectors K^α and the corresponding first degree invariants $K^\alpha p_\alpha$. Any Killing vector gives rise to a Killing tensor $K_{(\alpha} K_{\beta)}$ and to corresponding higher order generalizations. Such a Killing tensor is said to be reducible.

In this paper we will be mainly concerned with irreducible second rank Killing tensors. An important property of the Killing tensor equations is that they can be decomposed in conformal (traceless) part and trace parts. In the second rank case, the conformal part is

$$L_{(\alpha\beta;\gamma)} - \frac{1}{2} g_{(\alpha\beta} L^\mu{}_{\gamma);\mu} = 0 , \quad (13)$$

and the trace part is

$$K_{;\alpha} = -L^{\beta}_{\alpha;\beta} , \quad (14)$$

where the Killing tensor itself is decomposed in a conformal part $L_{\alpha\beta}$ and the trace $K = K^{\alpha}_{\alpha}$ according to

$$K_{\alpha\beta} = L_{\alpha\beta} + \frac{1}{n} K g_{\alpha\beta} . \quad (15)$$

Referring to the vector $L_{\alpha} = L^{\beta}_{\alpha;\beta}$ as the conformal current, it follows from (14) that, for any given conformal Killing tensor, the equation for the trace can be solved if the integrability condition

$$L_{[\alpha;\beta]} = 0 \quad (16)$$

is satisfied. The procedure to solve the Killing tensor equations is therefore to first solve the conformal Killing tensor equations and then check if the integrability condition (16) for the trace can be satisfied. In fact it turns out that the conformal equations can easily be solved leaving the integrability condition as the remaining equation to study.

2.2 Two dimensional systems and their quadratic invariants

A general 2-dimensional positive definite dynamical metric can then be written in the manifestly conformally flat form

$$ds^2 = 2G(x, y)(dx^2 + dy^2) , \quad (17)$$

where, comparing with (9), $G \equiv E - V$. It is simpler to work with complex coordinates,

$$\begin{aligned} z &= x + iy, & p_z &= p = \frac{1}{2}(p_x - ip_y), \\ \bar{z} &= x - iy, & p_{\bar{z}} &= \bar{p} = \frac{1}{2}(p_x + ip_y), \end{aligned} \quad (18)$$

so that

$$ds^2 = 2G(z, \bar{z})dzd\bar{z} . \quad (19)$$

The Jacobi Hamiltonian is

$$H_J = \frac{p\bar{p}}{G(z, \bar{z})} , \quad (20)$$

and the null Hamiltonian can be then written in the form

$$\mathcal{H} = 2p\bar{p} - G(z, \bar{z}) \equiv 0. \quad (21)$$

The equations of motion given by (21) are

$$\begin{aligned} \frac{dz}{dt} &= 2\bar{p}, \quad (+ \text{ c.c.}) \\ \frac{dp}{dt} &= G_{,z} = -V_{,z}, \quad (+ \text{ c.c.}), \end{aligned} \quad (22)$$

where c.c. stands for complex conjugate. G and V are always assumed to be real functions.

When doing calculations it is convenient to employ either an orthonormal frame $\omega^{\hat{I}}$ using hatted upper case Latin indices ($\hat{I}, \hat{J}, \hat{K}, \dots = \hat{1}, \hat{2}$) in terms of which the metric is written as

$$ds^2 = (\omega^{\hat{1}})^2 + (\omega^{\hat{2}})^2 , \quad (23)$$

or a complex null frame Ω^I using upper case Latin indices ($I, J, K, \dots = 1, 2$) with the metric written as

$$ds^2 = 2\Omega^1\Omega^2 . \quad (24)$$

Let $K_{\hat{I}\hat{J}}$ be the components of a 2nd rank Killing tensor with respect to the orthonormal frame $\omega^{\hat{I}}$ and let $e_I^{\hat{M}}$ be the matrix giving the transformation between the two frames. Then the null frame components are given by $K_{IJ} = e_I^{\hat{M}} e_J^{\hat{N}} K_{\hat{M}\hat{N}}$ which gives

$$\begin{aligned} K_{11} &= \frac{1}{2}(K_{\hat{1}\hat{1}} - K_{\hat{2}\hat{2}}) - iK_{\hat{1}\hat{2}} , \\ K_{22} &= \frac{1}{2}(K_{\hat{1}\hat{1}} - K_{\hat{2}\hat{2}}) + iK_{\hat{1}\hat{2}} , \\ K_{12} &= K/2 = \frac{1}{2}(K_{\hat{1}\hat{1}} + K_{\hat{2}\hat{2}}) . \end{aligned} \tag{25}$$

Since the orthonormal components $K_{\hat{I}\hat{J}}$ are by definition real it follows that $K_{22} = \overline{K_{11}}$. Therefore we can represent the null frame components of the conformal part of the Killing tensor by a single complex function $S(z, \bar{z})$ and the Killing tensor can be structured according to

$$(K_{MN}) = \begin{pmatrix} \bar{S}G & K/2 \\ K/2 & SG \end{pmatrix} , \tag{26}$$

the trace K being a real function. The conformal Killing tensor equations (13) are

$$G^{1/2} \bar{S}_{,z} = 0 , \quad G^{1/2} S_{,\bar{z}} = 0 . \tag{27}$$

It follows that K_{MN} is a conformal Killing tensor precisely if S is a function of z only. There is an infinite-dimensional family of conformal Killing tensors in this case parametrized by the single complex analytic function $S(z)$. One way of stating this result is that the equations (27) for the conformal Killing tensor coincide with Cauchy-Riemann's equations for certain rescaled linear combinations of the tracefree parts of the Killing tensor. This result was implicit in [20]. Thus the conformal Killing tensor in two dimensions has the form

$$(L_{MN}) = \begin{pmatrix} \bar{S}(\bar{z})G & 0 \\ 0 & S(z)G \end{pmatrix} . \tag{28}$$

The trace equations (14) can then be written in the form

$$\begin{aligned} K_{,z} &= -2\bar{S}(\bar{z})G_{,\bar{z}} - G\bar{S}'(\bar{z}) , \\ K_{,\bar{z}} &= -2S(z)G_{,z} - GS'(z) , \end{aligned} \tag{29}$$

the derivative $K_{,\bar{z}}$ being given by taking the complex conjugate of the above equation. We have now to write down the integrability condition for the conformal current:

$$L_{[1;2]} = -i \Im \{ 2G^{-1}G_{,zz}S(z) + 3G^{-1}G_{,z}S'(z) + S''(z) \} = 0 , \tag{30}$$

where $\Im$ denotes the imaginary part. This is exactly what we need to guarantee the existence of a 2nd rank Killing tensor.

We use a conformal transformation to standardize the frame and coordinate representation of the conformal Killing tensor. To that end we introduce the new null frame

$$\tilde{\Omega}^1 = B\Omega^1 , \tag{31}$$

$$\tilde{\Omega}^2 = B^{-1}\Omega^2 \tag{32}$$

and a new complex coordinate w by means of the transformation

$$z = F(w) . \tag{33}$$

By choosing

$$B = \sqrt{\bar{F}'(\bar{w})/F'(w)} \quad (34)$$

we ensure that the new frame $\tilde{\Omega}^I$ has the coordinate representation

$$\tilde{\Omega}^1 = \tilde{G}^{1/2} dw, \quad (35)$$

$$\tilde{\Omega}^2 = \tilde{G}^{1/2} d\bar{w} \quad (36)$$

where

$$\tilde{G} = |F'(w)|^2 G \quad (37)$$

is the new metric conformal factor, so that

$$ds^2 = 2\tilde{G}(w, \bar{w}) dw d\bar{w}. \quad (38)$$

It follows that the conformal Killing tensor components in the new frame are given by

$$\begin{aligned} \tilde{L}_{11} &= B^{-2} L_{11} = [\bar{F}'(\bar{w}(\bar{z}))]^{-2} \bar{S}(\bar{z}) \tilde{G}, \\ \tilde{L}_{22} &= B^2 L_{22} = [F'(w(z))]^{-2} S(z) \tilde{G}. \end{aligned} \quad (39)$$

The coordinates are standardized by choosing the conformal transformation function F so that

$$[F'(w(z))]^2 = S(z) \quad (40)$$

implying that $\tilde{L}_{11} = \tilde{L}_{22} = \tilde{G}$. Note that $\det(L_{IJ})$ is always positive. Therefore there is only one type of Killing tensors for positive definite Hamiltonians unlike the situation in the Lorentzian case. For a standardized conformal Killing tensor the integrability condition (30) simplifies to

$$\tilde{G}_{,ww} = \tilde{G}_{,\bar{w}\bar{w}}, \quad (41)$$

that is the wave equation. Writing $w = X + iY$, the solution of this equation is

$$\tilde{G} = A(X) + B(Y). \quad (42)$$

In virtue of (37), we note that the original potential is given in terms of the standardized potential by the relation

$$G = |S(z)|^{-1} \tilde{G}. \quad (43)$$

Putting together the last two formulas we get

$$G(X, Y) = \frac{A(X) + B(Y)}{|S(w)|}, \quad (44)$$

where the conformal factor is expressed in terms of the new coordinates. (44) is the usual form of the potential in separable coordinates.

In [14, 15] it is shown that also the conformal parts of higher rank Killing tensors are given by arbitrary analytic functions and the Killing tensor equations have been solved in the 3rd and 4th rank case respectively.

2.3 Invariants in standard and ordinary coordinates

The conformal transformation $z \rightarrow w = X + iY$ given by the holomorphic function (33) is such that

$$\begin{aligned} w &= F^{-1}(z) \\ P &= F'p, \quad (+ \text{ c.c.}), \end{aligned} \quad (45)$$

so that the Jacobi Hamiltonian now is

$$\tilde{H}_J = \frac{P\bar{P}}{\tilde{G}(w, \bar{w})} , \quad (46)$$

and (21) transforms into the new Hamiltonian

$$\tilde{\mathcal{H}} = \frac{2 P\bar{P} - \tilde{G}(w, \bar{w})}{|F'|^2} \equiv 0, \quad (47)$$

where

$$\tilde{G} = |F'|^2 G. \quad (48)$$

If the choice (40) is made, (48) agrees with (43). Let us introduce the “standard” null Hamiltonian

$$\mathcal{H}_S = 2 P\bar{P} - \tilde{G}(w, \bar{w}) \equiv 0, \quad (49)$$

that implies the new “time” s such that

$$\frac{d}{ds} = |F'|^2 \frac{d}{dt}. \quad (50)$$

Equations of motion given by (49) are then

$$\begin{aligned} \frac{dw}{ds} &= 2\bar{P}, \quad (+ \text{ c.c.}) \\ \frac{dP}{ds} &= \tilde{G}_{,w} \quad (+ \text{ c.c.}). \end{aligned} \quad (51)$$

It is easy to check that (22) and (51) are equivalent. At the same time, a conserved quantity stays conserved if transformed between the two gauges (47) and (49).

In the standardized frame, the invariant $I_J = K^{MN} p_M p_N$ is simply given by

$$I_J = P^2 + \bar{P}^2 + K(w, \bar{w}) \tilde{H}_J. \quad (52)$$

Interpreting the Jacobi Hamiltonian as the geometric energy constant, equal simply to one half for every geodesic, the passage from the geometric gauge to the physical gauge is accomplished by the recipe of performing the substitutions

$$\tilde{H}_J \mapsto \frac{1}{2}, \quad H_J \mapsto \frac{1}{2} \quad (53)$$

wherever the Jacobi Hamiltonians appear. The invariant can therefore be written as

$$I = P^2 + \bar{P}^2 + \frac{1}{2} K(w, \bar{w}). \quad (54)$$

The subscript J is used to distinguish the invariant in the Jacobi time gauge from the invariant I in the physical time gauge. The trace equation [see (14)] in the separating coordinates can be written

$$\begin{aligned} K_{,X} &= -2\tilde{G}_{,X} , \\ K_{,Y} &= 2\tilde{G}_{,Y} , \end{aligned} \quad (55)$$

with the solution

$$K = -2A(X) + 2B(Y). \quad (56)$$

We wish to express the invariant in terms of the coordinate momentum components. The null frame components are related to the complex coordinate components by $(p_1, p_2) = G^{-1/2}(p_z, p_{\bar{z}}) = G^{-1/2}(p, \bar{p})$. The invariant can then be written in complex coordinates as

$$I_J = S(z)p^2 + \bar{S}(\bar{z})\bar{p}^2 + G^{-1}Kp\bar{p} . \quad (57)$$

Using relation (18) we obtain the invariant in real coordinates as

$$I_J = \frac{1}{2}\Re(S)(p_x^2 - p_y^2) + \Im(S)p_x p_y + \frac{1}{4}G^{-1}K(p_x^2 + p_y^2) . \quad (58)$$

Note that the last term is equal to KH_J so that, applying recipe (53) and using solution (56), implies

$$I = \frac{1}{2}\Re(S)(p_x^2 - p_y^2) + \Im(S)p_x p_y - A(X(x, y)) + B(Y(x, y)) . \quad (59)$$

3 Separating coordinate systems

The integrability condition (30) can be interpreted as follows. Given any analytic function $S(z)$ there is a family of potentials (the solutions of (30)) which is integrable at zero energy. A common situation is that one is interested in invariants which are valid for arbitrary values of the energy. Invariants of that type arise if the integrability condition itself is independent of the energy. This happens precisely if

$$\Im\{S''(z)\} = 0 . \quad (60)$$

Since $S(z)$ is an analytic function, it follows that $S''(z)$ is a real constant and therefore $S(z)$ must be a second degree polynomial

$$S(z) = S_2(z) := az^2 + \beta z + \gamma , \quad (61)$$

where a is a real constant and β and γ are complex constants. The integrability condition (30) then simplifies to

$$\Im\{2G_{,zz}S_2(z) + 3G_{,z}S_2'(z)\} = 0 . \quad (62)$$

This is nothing but Darboux's condition for quadratic constants of the motion [12] written in complex variables. The general integrability condition (30) therefore generalizes Darboux's condition to include also quadratic invariants at fixed energy.

In the arbitrary energy case we then have that (40) becomes $S_2(z) = [F'(w(z))]^2$. This equation can then be integrated resulting in (without loss of generality we may put $a = 1$ if $a \neq 0$ since (62) is invariant under a real scaling of S)

$$w = \begin{cases} \log(\sqrt{S_2} + z + \beta/2) + const , & (a = 1) , \\ 2\beta^{-1}\sqrt{\beta z + \gamma} + const , & (a = 0, \beta \neq 0) , \\ \gamma^{-1/2}z + const , & (a = 0, \beta = 0) . \end{cases} \quad (63)$$

Inverting these relations we must distinguish between the four cases

$$\begin{aligned} \text{(i)} \quad & a = 1, \Delta := \sqrt{\gamma - \beta^2/4} \neq 0 , \\ \text{(ii)} \quad & a = 1, \Delta = 0 , \\ \text{(iii)} \quad & a = 0, \beta \neq 0 , \\ \text{(iv)} \quad & a = 0, \beta = 0 , \end{aligned} \quad (64)$$

giving rise to the following explicit formulas for $F(w)$

$$z = F(w) = \begin{cases} \Delta \sinh(w - w_0) - \beta/2 , & \text{(i)} \\ e^{w-w_0} - \frac{\beta}{2} , & \text{(ii)} \\ \frac{1}{4}\beta(w - w_0)^2 - \gamma/\beta , & \text{(iii)} \\ \gamma^{1/2}w - w_0 . & \text{(iv)} \end{cases} \quad (65)$$

where w_0 is an arbitrary constant. We are primarily interested in proper conformal transformation, i.e. $S(z) \neq 1$. Therefore we are free to perform translations and rotations in z and w to simplify formulas. In particular we may put w_0 equal to zero by a translation of the origin in the w -plane. We can also rotate the

z -plane to transform Δ to a positive real number in case (i). Likewise β may be assumed real in case (iii) and γ can be taken as real and positive in case (iv). The argument we used to set $a = 1$ can then be used to set $\beta = 4$ for convenience in case (iii). We also translate the z -origin obtaining finally

$$F(w) = \begin{cases} \Delta \sinh w , & \text{(i)} \\ e^w , & \text{(ii)} \\ w^2 , & \text{(iii)} \\ \gamma^{1/2} w , & \text{(iv)} \end{cases} \quad (66)$$

where Δ is now to be understood as a positive real number. Differentiation of the expressions (66) yields

$$F'(w) = \begin{cases} \Delta \cosh w , & \text{(i)} \\ e^w , & \text{(ii)} \\ 2w , & \text{(iii)} \\ \gamma^{1/2} , & \text{(iv)} \end{cases} \quad (67)$$

implying that the conformal transformation factors take the forms

$$|S(z)| = |F'(w)|^2 = \begin{cases} \Delta^2(\sinh^2 X + \cosh^2 Y) = \sqrt{(r^2 + \Delta^2)^2 - 4\Delta^2 x^2} , & \text{(i)} \\ e^{2X} = r^2 , & \text{(ii)} \\ 4(X^2 + Y^2) = 4r , & \text{(iii)} \\ |\gamma| , & \text{(iv)} \end{cases} \quad (68)$$

where $r := \sqrt{x^2 + y^2}$.

Cases (i)–(iv) correspond to all the separating coordinate systems in two dimensions and are in agreement with the theory of Stäckel's classical integrable potentials.

In case (i) the conformal coordinate transformation is given by

$$\begin{aligned} x &= \Delta \sinh X \cos Y , \\ y &= \Delta \cosh X \sin Y . \end{aligned} \quad (69)$$

This is the transformation used to define the Stäckel potential in *elliptical coordinates* which was also discussed by Darboux [12]. From (68) and (43) we find that the potential is given by the well-known formula [13]

$$G(\lambda, \mu) = \frac{A(\lambda) + B(\mu)}{\lambda - \mu} , \quad (70)$$

where

$$\begin{aligned} \lambda + \alpha &:= 2\Delta^2 \sinh^2 X = r^2 - \Delta^2 + \sqrt{(r^2 + \Delta^2)^2 - 4\Delta^2 x^2} , \\ \mu + \beta &:= 2\Delta^2 \sin^2 Y = -r^2 + \Delta^2 + \sqrt{(r^2 + \Delta^2)^2 - 4\Delta^2 x^2} \end{aligned} \quad (71)$$

$\Delta^2 = \beta - \alpha$ with α and β constants and the functions A and B are arbitrary functions of their arguments.

The case (ii) conformal transformation is given by

$$\begin{aligned} x &= e^X \cos Y , \\ y &= e^X \sin Y . \end{aligned} \quad (72)$$

In particular it follows that $Y = \arctan(y/x) =: \vartheta$ is a polar angle. Using (68) it then follows that the potential can be written as

$$G(x, y) = A(r) + r^{-2} B(\vartheta) , \quad (73)$$

where A and B are arbitrary functions. This is the case of integrability in *polar coordinates*.

Case (iii) is characterized by the conformal transformation

$$\begin{aligned} x &= X^2 - Y^2 , \\ y &= 2XY . \end{aligned} \quad (74)$$

case	conformal transformation $w = F^{-1}(z)$	conformal Killing tensor component $S(z) = [F'(w(z))]^2$	coordinate system
(i)	$\ln z \pm \sqrt{z^2 - \Delta^2}$	$z^2 - \Delta^2$	elliptical
(ii)	$\ln z$	z^2	polar
(iii)	$\sqrt{z}$	$4z$	parabolic
(iv)	$\gamma^{-1/2}$	γ	affine Cartesian

Table 1: Conformal transformation functions giving rise to systems with a second invariant which is linear or quadratic in the momenta.

Using (68) this gives immediately the likewise well-known classical integrable potential

$$G(x, y) = [A(r + x) + B(r - x)]/r , \quad (75)$$

where again A and B are arbitrary functions. This case correspond to separability in *parabolic coordinates*

$$\begin{aligned} X &= \sqrt{\frac{r+x}{2}} , \\ Y &= \sqrt{\frac{r-x}{2}} . \end{aligned} \quad (76)$$

Finally in case (iv) the conformal transformation factor is a constant. In the trivial case $\gamma = 1$, we have separability in Cartesian coordinates so the potential is simply given by the explicitly separated form

$$G(x, y) = A(x) + B(y) , \quad (77)$$

in terms of the arbitrary functions A and B . A general complex value of γ provides a linear transformation of the plane, including rotations.

This completes the list of the four classical cases. The four nontrivial conformal transformations giving rise to systems with a second linear or quadratic invariant are tabulated in table 1.

Summarizing the results obtained so far, if we construct the conformal transformation by means of the holomorphic function $S(z)$ according to relation (40) or, equivalently, such that

$$F'(w(z)) = \sqrt{S(z)} \quad (78)$$

and the S function satisfies condition (60), the Hamiltonian is actually separable for every value of the energy, that is the Hamilton-Jacobi equation

$$\mathcal{H}(\mathcal{W}_w, \mathcal{W}_{\bar{w}}, w, \bar{w}) = 0, \quad (79)$$

for the Hamilton principal function $\mathcal{W}(w, \bar{w})$, can be solved by separation of variables. This is equivalent to state that the allowed coordinate transformation are those and only those such that the conformal factor can be written in the form

$$|F'|^2 = f_X \left(\frac{w + \bar{w}}{2} \right) + f_Y \left(\frac{w - \bar{w}}{2i} \right). \quad (80)$$

4 Main result

We can now state the

Theorem 1: *Let a Hamiltonian system with two degrees of freedom be defined by Hamiltonian function (49). Then the following statements are equivalent:*

(1). *The Hamiltonian vector field corresponding to (49)*

$$\mathbf{V} = 2 \Re\{2\bar{P}\partial_w + (\partial_w\tilde{G})\partial^w\} \quad (81)$$

admits the bi-Hamiltonian representation defined by the constant Poisson bi-vectors

$$\mathcal{P}_0 = \partial_w \wedge \partial^w + \partial_{\bar{w}} \wedge \partial^{\bar{w}} = 2 \Re\{\partial_w \wedge \partial^w\} \quad (82)$$

and

$$\mathcal{P}_1 = \partial_w \wedge \partial^{\bar{w}} + \partial_{\bar{w}} \wedge \partial^w = 2 \Re\{\partial_w \wedge \partial^{\bar{w}}\}, \quad (83)$$

such that

$$\mathbf{V} = [\mathcal{P}_0, \mathcal{H}_S] = [\mathcal{P}_1, I], \quad (84)$$

where I is the “Hamiltonian” 54.

(2). *$\tilde{G}$ satisfies the wave equation (41).*

(3). *The dynamics given by (49) can be described by the Lax equation*

$$\frac{dL}{ds} = [L, M] = LM - ML, \quad (85)$$

where the matrices L and M constitutes the following Lax pair:

$$L = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}, \quad M = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} \quad (86)$$

where

$$L_1 = \begin{pmatrix} P + \bar{P} & w + \bar{w} \\ -\frac{2A}{w+\bar{w}} & -P - \bar{P} \end{pmatrix}, \quad (87)$$

$$L_2 = i \begin{pmatrix} P - \bar{P} & \bar{w} - w \\ \frac{2B}{\bar{w}-w} & \bar{P} - P \end{pmatrix}, \quad (88)$$

and

$$M_1 = \frac{1}{w + \bar{w}} \begin{pmatrix} 0 & 0 \\ \frac{d}{ds}(P + \bar{P}) & 2(P + \bar{P}) \end{pmatrix} \quad (89)$$

$$M_2 = \frac{1}{w - \bar{w}} \begin{pmatrix} 0 & 0 \\ \frac{d}{ds}(\bar{P} - P) & 2(\bar{P} - P) \end{pmatrix} \quad (90)$$

for some $A, B \in C^1(\mathbf{R})$.

Proof: The proof goes along the lines of paper [8], that is $(1) \rightarrow (2) \rightarrow (3) \rightarrow (1)$.

$(1) \rightarrow (2)$. Let the Hamiltonian system defined by Hamiltonian function (49) admits a second Hamiltonian representation with respect to the Poisson bi-vector $\mathcal{P}_1$ defined by eq. (83). Then, obviously, the flow of $\mathbf{V}$ preserves both $\mathcal{P}_0$ and $\mathcal{P}_1$:

$$\mathcal{L}_{\mathbf{V}}(\mathcal{P}_0) = \mathcal{L}_{\mathbf{V}}(\mathcal{P}_1) = 0, \quad (91)$$

where

$$\mathcal{L}_{\mathbf{V}}(\cdot) = [\mathbf{V}, \cdot] \quad (92)$$

denotes Lie derivation along the vector field X . Now, taking into account (81), we compute

$$0 = \mathcal{L}_{\mathbf{V}}(\mathcal{P}_1) = (\partial_{\bar{w}\bar{w}}\tilde{G} - \partial_{ww}\tilde{G})\partial_w \wedge \partial^{\bar{w}} \quad (93)$$

and the implication follows.

(2) $\rightarrow$ (3). Assume the “potential” $\tilde{G}$ in (49) satisfies eq. (41). The Hamiltonian can then be written as

$$\mathcal{H}_S = 2 P \bar{P} - A \left(\frac{w + \bar{w}}{2} \right) - B \left(\frac{w - \bar{w}}{2i} \right). \quad (94)$$

The corresponding HVF is

$$\mathbf{V} = 2\bar{P}\partial_w + 2P\partial_{\bar{w}} + \partial_w(A+B)\partial^w + \partial_{\bar{w}}(A+B)\partial^{\bar{w}}, \quad (95)$$

so that function (54), namely

$$I = P^2 + \bar{P}^2 + B - A, \quad (96)$$

is a second constant of the motion. We can introduce the two functions

$$\sigma_{1,2} = I \pm \mathcal{H}_S = (P \pm \bar{P})^2 + B - A \mp (A + B). \quad (97)$$

Hence, for the matrices (87) and (88), the corresponding characteristic polynomials are respectively

$$\lambda^2 + 2A - (P + \bar{P})^2 = 0 \quad (98)$$

and

$$\mu^2 + 2B + (P - \bar{P})^2 = 0 \quad (99)$$

with eigenvalues

$$\lambda_{\pm} = \pm\sqrt{\sigma_1}, \quad \sigma_1 = I + \mathcal{H}_S = (P + \bar{P})^2 - 2A \quad (100)$$

and

$$\mu_{\pm} = \pm\sqrt{\sigma_2}, \quad \sigma_2 = I - \mathcal{H}_S = (P - \bar{P})^2 + 2B \quad (101)$$

and with eigenvectors

$$\mathbf{E}_1^{\pm} = \begin{pmatrix} 1 \\ m_1^{\pm} \end{pmatrix}, \quad m_1 = \frac{\lambda_{\pm} - P - \bar{P}}{w + \bar{w}} \quad (102)$$

and

$$\mathbf{E}_2^{\pm} = \begin{pmatrix} 1 \\ m_2^{\pm} \end{pmatrix}, \quad m_2 = \frac{\mu_{\pm} - P + \bar{P}}{\bar{w} - w} \quad (103)$$

By direct inspection, the matrices (89) and (90) satisfies the matrix equation (85) which describes the dynamics (51). Moreover, verifying that

$$M_k = \frac{d}{ds}(R_k)R_k^{-1}, \quad k = 1, 2, \quad (104)$$

where

$$R_k = \begin{pmatrix} 1 & 1 \\ m_k^+ & m_k^- \end{pmatrix}, \quad (105)$$

one can easily verify that $L(s)$ and $L(0)$ are unitary equivalent so that the matrix equation (85) is indeed a Lax representation of the Hamiltonian system (49) and (51).

(3) $\rightarrow$ (1). Assume that point (3) is true. From the dynamically invariant eigenvalues (100) and (101), the first integrals (49) and (96) are expressed as

$$\mathcal{H}_S = \frac{1}{2}(\sigma_1 - \sigma_2), \quad I = \frac{1}{2}(\sigma_1 + \sigma_2). \quad (106)$$

To grant the validity of (84), with the HVF given by (81), it is easy to reconstruct the structure of the two Poisson bi-vectors that turn out to be of the forms (82) and (83). End of the proof.

5 Remarks and conclusions

We remark that theorem 1 can also be stated in terms of real variables, but with the above presentation the parallel with the work of McLenaghan and Smirnov [8] is complete. Our result is however more general, even considering the paper of Bruce et al. [17] To place the subclass of systems considered by them in the general setting presented above, it is sufficient to recall that the change of variables

$$X = x - y, \quad Y = x + y, \quad (107)$$

which stays at the hearth of their treatment, corresponds to the conformal transformation (iv) of Section 3 with $\gamma = \frac{i}{2}$.

We agree with the remark that, interpreting the construction of theorem 1 as a Poisson-Nijenhuis structure on the cotangent space, does not provide a system of conserved quantities, but this has no practical consequences in the present context. It remains an open question if the corresponding treatment of higher-dimensional systems strictly requires the extension to a Gel'fand-Zakharevich structure [5].

We conclude mentioning the possibility of further extensions of this picture to integrable systems admitting more involved forms for the second invariant. Very few examples of explicitly solvable non-separable systems are known so that the discovery of bi-Hamiltonian structures admitting higher-order or transcendental invariants could have far reaching consequences in the study of integrable systems.

References

- [1] S. Rauch-Wojciechowski, Phys. Lett. 160A (1991) 149–154.
- [2] M. Antonowicz, S. Rauch-Wojciechowski, Phys. Lett. 171A (1992) 303–309.
- [3] S. Benenti, Intrinsic characterization of the variable separation in the Hamilton-Jacobi equation, J. Math. Phys. 38 (1997) 6578–6602.
- [4] C. Morosi and G. Tondo, Quasi-bihamiltonian systems and separability. J. Phys. A 30 (1997) 2799–2806.
- [5] A. Ibort, F. Magri, G. Marmo, Bihamiltonian structures and Stäckel separability, J. Geom. Phys. 33 (2000) 210–228.
- [6] M. Blaszak, Bi-Hamiltonian separable chains on Riemannian manifolds, Phys. Lett. A 243 (1998) 25–32.
- [7] G. Falqui and M. Pedroni, preprint nlin.SI/0204029 (2002)
- [8] R. G. McLenaghan, R. G. Smirnov, A class of Liouville-integrable Hamiltonian systems with two degrees of freedom, J. Math. Phys. 41 (2000) 6879–6889.

- [9] V. I. Arnold, *Mathematical methods of classical mechanics* (Springer-Verlag, New York, 1978).
- [10] C. Lanczos, *The variational principles of mechanics* (Dover, New York, 1986).
- [11] K. Rosquist, G. Pucacco, Invariants at fixed and arbitrary energy. A unified geometric approach, J. Phys. A 28 (1995) 3235–3252 (RP).
- [12] G. Darboux, Sur un problème de mécanique, Archives Néerlandaises (ii) 6 (1901) 371–377.
- [13] J. Hietarinta, Direct methods for the search of the second invariant Phys. Rep. 147 (1987) 87–154.
- [14] M. Karlovini, K. Rosquist, A unified treatment of cubic invariants at fixed and arbitrary energy, J. Math. Phys. 41 (2000) 370–384.
- [15] M. Karlovini, G. Pucacco, K. Rosquist, L. Samuelson, A unified treatment of quartic invariants at fixed and arbitrary energy, J. Math. Phys. 43 (2002) 4041–4059.
- [16] B. A. Dubrovin, A. T. Fomenko, S. P. Novikov, *Modern Geometry – Methods and applications* (Part I. Second edition, Springer-Verlag, New York, 1992).
- [17] R. G. Bruce, R. G. McLenaghan, R. G. Smirnov, A geometric approach to the problem of integrability of Hamiltonian systems by separation of variables, J. Geom. Phys. 39 (2001) 301–322
- [18] R. Abraham, J. E. Marsden, *Foundations of mechanics* (Benjamin Cummings, New York, 1978)
- [19] B. F. Schutz, *Geometrical methods of mathematical physics* (Cambridge University Press, Cambridge, U.K., 1980).
- [20] V. N. Kolokol'tsov, Geodesic flows on two-dimensional manifolds with an additional first integral that is a polynomial in the velocities, Math. USSR Izvestiya, 21, (1983) 291–306.