An algebraic method to compute the critical points of the distance function between two Keplerian orbits

Giovanni F. Gronchi
Department of Mathematics, University of Pisa, largo Pontecorvo 5, 56127 Pisa, Italy
e-mail: gronchi@dm.unipi.it

Abstract. We describe an efficient algorithm to compute all the critical points of the distance function between two Keplerian orbits (either bounded or unbounded) with a common focus. The critical values of this function are important for different purposes, for example to evaluate the risk of collisions of asteroids or comets with the Solar system planets. Our algorithm is based on the algebraic elimination theory: through the computation of the resultant of two bivariate polynomials, we find a 16^{th} degree univariate polynomial whose real roots give us one component of the critical points. We discuss also some degenerate cases and show several examples, involving the orbits of the known asteroids and comets.

Keywords: collisions, asteroids and comets, MOID, algebraic methods

ἐϑεώρουν σε σπεύδοντα μετασχεῖν
τῶν πεπραγμένων ἡμῖν κωνικῶν [1]
(Apollonius of Perga, *Conics*, Book I)

1. Introduction

The mutual position of two osculating Keplerian orbits with a common focus can give us interesting information on the possibility of collisions or close approaches between two celestial bodies that follow approximately these trajectories. As it is well known these orbits, solutions to the Kepler problem, are *conics*, either bounded (circles, ellipses) or unbounded (parabolas, hyperbolas).

Given two Keplerian orbits, it is particularly interesting to determine the Minimal Orbital Intersection Distance (MOID), that is the absolute minimum of the Euclidean distance d between a point on the first orbit and a point on the second one. Indeed the square of this distance d^2 is always used, to have a smooth function of the angular variables on the orbits also when the MOID is zero. In this way we can compute the MOID by searching for all the *critical points* (or *stationary points*) of the squared distance d^2 and then selecting the minimum among

[1] I observed you were quite eager to be kept informed of the work I was doing in conics.

the values at those points, that are finitely many for orbits in generic position.

There are several papers available in the literature that deal with the computation of the MOID; see for example (Sitarski, 1968), (Hoots, 1984), (Dybczynski et al., 1986). The main difficulty in the algorithms proposed by these authors is to deal with a nonlinear one-dimensional equation appearing when we solve for a component of the critical points of d^2.

Recently, for the case of two elliptic orbits, the equations of the critical points of d^2 have been interpreted as a polynomial system and some algebraic geometry methods have been exploited to compute all of its solutions. In (Kholshevnikov and Vassiliev, 1999) Gröbner bases theory has been used to obtain a trigonometric polynomial whose real roots represent one component of all the solutions. In (Gronchi, 2002) an algorithm is introduced, based on the resultant theory (Cox et al., 1992) and the Fast Fourier Transform (FFT) to perform the elimination of one variable; an upper bound on the maximum number of critical points (if they are finitely many) is also obtained by using Newton's polytopes and Bernstein's theorem (Bernstein, 1975).

The use of algebraic elimination methods, that generalize Gauss' elimination procedure from linear to nonlinear polynomial equations, turns out to be a powerful tool to deal with this problem, avoiding for example all the troubles that may arise when searching for a good *starting guess* of Newton's method.

In (Gronchi, 2002) we also stress the importance of computing all the stationary points of d^2: in fact there are cases, with orbits of NEAs and of the Earth, for which a low value of the distance d can be attained also at different local minima, and even at saddle points.

The use of the eccentric anomaly, as in both (Kholshevnikov and Vassiliev, 1999) and (Gronchi, 2002), simplifies the formulas, but introduces for $e = 1$ an artificial singularity; this can be avoided by using the true anomalies, as is done in (Sitarski, 1968), where the algorithm was conceived just to compute the MOID of the comets with respect to the outer planets orbits.

In this paper, by using the true anomalies as orbital parameters, we generalize the method presented in (Gronchi, 2002) to all the Keplerian orbits (including parabolas and hyperbolas). Furthermore we add several improvements to our previous work, which are also important from the computational point of view:

1. the mutual variables, useful to understand the effective dimensionality of the problem, are singular for vanishing mutual inclination, therefore in this paper we use the two complete sets of orbital

elements. However, when we perform large scale numerical experiments, we can use the mutual variables to produce different orbital configurations in terms of the Keplerian elements (see Section 8);

2. by an appropriate manipulation of the Sylvester matrix (see Subsections 4.2, 10.2) we are able to factorize the resultant polynomial and to obtain a 16^{th} degree univariate polynomial, whose real roots represent one component of the critical points;

3. due to the lower degree of the univariate polynomial, we easily succeed in applying the FFT methods (that optimally work with a number of evaluations that is a power of 2) using only $16 = 2^4$ polynomial evaluations instead of $32 = 2^5$, as in our previous work.

We observe that in (Kholshevnikov and Vassiliev, 1999) an 8^{th} degree trigonometric polynomial $g(u)$ (function of $\sin u, \cos u$) is computed, that plays the same role of our 16^{th} degree polynomial, anyway their method requires a symbolic manipulation program to perform the elimination. In this paper we shall make a self contained computation of this 16^{th} degree polynomial; furthermore, as it will be obtained as the determinant of a matrix, we shall directly work on the coefficients of this matrix, that are polynomials of lower degree.

In Sections 2, 3 and 4 we introduce the problem, its algebraic formulation and our algorithm to solve it. In Section 5 we present a useful improvement to the algorithm: we use an angular shift along the elliptic orbits to control the size of the roots of the polynomial equations that we are solving and to avoid sending roots to infinity. In Sections 6, 7 we shall study some properties of the critical points: in particular we shall estimate the size of their corresponding anomalies in the case of parabolic and hyperbolic orbits, and we shall characterize the cases with infinitely many critical points. In Section 8 we shall present some examples with a high number of critical points and some applications to Solar system orbits.

2. Critical points of the squared distance

Let us consider two Keplerian orbits with a common focus. We shall use the cometary elements $(Q, E, i_1, \Omega_1, \omega_1, V)$ and $(q, e, i_2, \Omega_2, \omega_2, v)$ to describe these orbits, that are respectively *perihelion distance, eccentricity, inclination, longitude of perihelion, perihelion argument* and *true anomaly.* The orbits, on their respective planes, can be parametrized as follows

$$\begin{cases} X &= R\cos V \\ Y &= R\sin V \end{cases} \qquad \begin{cases} x &= r\cos v \\ y &= r\sin v \end{cases}$$

where

$$R = \frac{P}{1 + E\cos V}\,; \qquad r = \frac{p}{1 + e\cos v}\,;$$

and $P = Q(1+E), p = q(1+e)$ are the *conic parameters*.

Following (Sitarski, 1968) we can write the components of the orbits $\mathcal{X}_1 = (X_1\,, Y_1\,, Z_1)\,, \mathcal{X}_2 = (x_2\,, y_2\,, z_2)$ as

$$\mathcal{X}_1 = X\,\mathcal{P} + Y\,\mathcal{Q} = R\,[\mathcal{P}\cos V + \mathcal{Q}\sin V]\ ;$$

$$\mathcal{X}_2 = x\,\mathfrak{p} + y\,\mathfrak{q} = r\,[\mathfrak{p}\cos v + \mathfrak{q}\sin v]\ ;$$

with

$$\mathcal{P} = (P_x\,, P_y\,, P_z)\ ; \qquad \mathcal{Q} = (Q_x\,, Q_y\,, Q_z)\ ;$$
$$\mathfrak{p} = (p_x\,, p_y\,, p_z)\ ; \qquad \mathfrak{q} = (q_x\,, q_y\,, q_z)\ ;$$

where [2]

$$\begin{aligned}
&P_x = \cos\omega_1\,; \quad P_y = \sin\omega_1\cos i_1\,; \quad P_z = \sin\omega_1\sin i_1\,;\\
&Q_x = -\sin\omega_1\,; \quad Q_y = \cos\omega_1\cos i_1\,; \quad Q_z = \cos\omega_1\sin i_1\,;\\
&p_x = \cos\omega_2\cos(\Omega_2-\Omega_1) - \sin\omega_2\cos i_2\sin(\Omega_2-\Omega_1)\,;\\
&p_y = \cos\omega_2\sin(\Omega_2-\Omega_1) + \sin\omega_2\cos i_2\cos(\Omega_2-\Omega_1)\,;\\
&p_z = \sin\omega_2\sin i_2\,;\\
&q_x = -\sin\omega_2\cos(\Omega_2-\Omega_1) - \cos\omega_2\cos i_2\sin(\Omega_2-\Omega_1)\,;\\
&q_y = -\sin\omega_2\sin(\Omega_2-\Omega_1) + \cos\omega_2\cos i_2\cos(\Omega_2-\Omega_1)\,;\\
&q_z = \cos\omega_2\sin i_2\ .
\end{aligned}$$

REMARK: The following relations hold:

$$\|\mathcal{P}\| = \|\mathcal{Q}\| = \|\mathfrak{p}\| = \|\mathfrak{q}\| = 1\,; \qquad \langle\mathcal{P},\mathcal{Q}\rangle = \langle\mathfrak{p},\mathfrak{q}\rangle = 0\,;$$

where $\langle,\rangle$ is the Euclidean scalar product.

[2] The quantities $P_x, P_y, P_z, Q_x, Q_y, Q_z, p_x, p_y, p_z, q_x, q_y, q_z$ are the elements in the first two rows of the matrices

$$H_1 = \begin{bmatrix} 1 & 0 & 0\\ 0 & \cos i_1 & -\sin i_1\\ 0 & \sin i_1 & \cos i_1 \end{bmatrix}\begin{bmatrix} \cos\omega_1 & -\sin\omega_1 & 0\\ \sin\omega_1 & \cos\omega_1 & 0\\ 0 & 0 & 1 \end{bmatrix}$$

and

$$H_2 = \begin{bmatrix} \cos(\Omega_2-\Omega_1) & -\sin(\Omega_2-\Omega_1) & 0\\ \sin(\Omega_2-\Omega_1) & \cos(\Omega_2-\Omega_1) & 0\\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0\\ 0 & \cos i_2 & -\sin i_2\\ 0 & \sin i_2 & \cos i_2 \end{bmatrix}\begin{bmatrix} \cos\omega_2 & -\sin\omega_2 & 0\\ \sin\omega_2 & \cos\omega_2 & 0\\ 0 & 0 & 1 \end{bmatrix}$$

that are used to place the orbits in the 3-dimensional space. In (Sitarski, 1968) the same quantities are described as components of *cracovians*, that are ordinary matrices with a different multiplication rule. The cracovian calculus has been introduced by the Polish mathematician T. Banachiewicz; see (Banachiewicz, 1955).

The squared distance d^2 between two points on the two orbits is given by

$$d^2(V,v) = \langle \mathcal{X}_1 - \mathcal{X}_2, \mathcal{X}_1 - \mathcal{X}_2 \rangle \tag{1}$$

and we can write the equations for the stationary points of d^2 as

$$\begin{cases} \dfrac{R}{P}\,[ERY + Y(Kx+My) - (ER+X)(Lx+Ny)] = 0 \\[2ex] \dfrac{r}{p}\,[ery + y(KX+LY) - (er+x)(MX+NY)] = 0 \end{cases} \tag{2}$$

where

$$K = \langle \mathcal{P}, \mathfrak{p} \rangle\,; \quad L = \langle \mathcal{Q}, \mathfrak{p} \rangle\,; \quad M = \langle \mathcal{P}, \mathfrak{q} \rangle\,; \quad N = \langle \mathcal{Q}, \mathfrak{q} \rangle\ .$$

We rewrite system (2) by collecting its terms as follows:

$$\begin{cases} p\,[1+E\cos V]\Big\{\sin V\,[K\cos v + M\sin v] - \\ \quad - [E+\cos V]\,[L\cos v + N\sin v]\Big\} + EP\sin V[1+e\cos v] = 0\,; \\[2ex] P\,[1+e\cos v]\Big\{\sin v\,[K\cos V + L\sin V] - \\ \quad - [e+\cos v]\,[M\cos V + N\sin V]\Big\} + ep\sin v[1+E\cos V] = 0\ . \end{cases} \tag{3}$$

REMARK: We search for the *real solutions* of system (3). If $E \geq 1$ (resp. $e \geq 1$) we take only the solutions for which $1 + E\cos V > 0$ (resp. $1 + e\cos v > 0$).

REMARK: The values of the pairs (V, v) such that $1 + E\cos V = 1 + e\cos v = 0$ are always solutions of system (3): they are not real solutions if $E < 1$ or $e < 1$; otherwise they are real, but they have to be discarded because their components coincide with the angular value of the asymptote of the corresponding hyperbola, or with the value $-\pi$ if the orbit is parabolic.

3. Algebraic formulation of the problem

Following (Gronchi, 2002) we use the variable change

$$\begin{cases} s &= \tan(V/2) \\ t &= \tan(v/2) \end{cases} \tag{4}$$

to transform the problem into an algebraic one. Taking into account the relations

$$1 + E\cos V = \frac{(E+1) - s^2(E-1)}{1+s^2}\,; \;\; E + \cos V = \frac{(E+1) + s^2(E-1)}{1+s^2}\,;$$

$$1 + e\cos v = \frac{(e+1) - t^2(e-1)}{1+t^2}\,; \qquad e + \cos v = \frac{(e+1) + t^2(e-1)}{1+t^2}\,;$$

we have to solve the polynomial system

$$\begin{cases} f(s,t) &= f_4(t)\,s^4 + f_3(t)\,s^3 + f_2(t)\,s^2 + f_1(t)\,s + f_0(t) = 0 \\ g(s,t) &= g_2(t)\,s^2 + g_1(t)\,s + g_0(t) = 0 \end{cases} \tag{5}$$

with

$$\begin{aligned} f_0(t) &= p(E+1)^2(Lt^2 - 2Nt - L)\,; \\ f_1(t) &= -2\,[Kp(E+1) + EP(e-1)]\,t^2 + 4pM(E+1)\,t + \\ &\quad +2\,[Kp(E+1) + EP(e+1)]\;; \\ f_2(t) &= 0\,; \\ f_3(t) &= 2\,[Kp(E-1) - EP(e-1)]\,t^2 - 4pM(E-1)\,t - \\ &\quad -2\,[Kp(E-1) - EP(e+1)]\;; \\ f_4(t) &= -p(E-1)^2(Lt^2 - 2Nt - L) = -\frac{(E-1)^2}{(E+1)^2}\,f_0(t)\,; \end{aligned}$$

and

$$\begin{aligned} g_0(t) &= PM(e-1)^2\,t^4 + [-2KP(e-1) + 2ep(E+1)]\,t^3 + \\ &\quad + [2KP(e+1) + 2ep(E+1)]\,t - PM(e+1)^2\,; \\ g_1(t) &= 2PN(e-1)^2\,t^4 - 4PL(e-1)t^3 + 4PL(e+1)\,t - 2PN(e+1)^2\,; \\ g_2(t) &= -PM(e-1)^2\,t^4 + [2KP(e-1) - 2ep(E-1)]\,t^3 + \\ &\quad + [-2KP(e+1) - 2ep(E-1)]\,t + PM(e+1)^2\;. \end{aligned}$$

REMARK: The variable change (4) does not allow to take into account the values $V = \pi$ and $v = \pi$, that are sent to infinity: we have to take care of this fact when we deal with elliptic or circular orbits. A solution to this problem is the subject of Section 5.

4. Description of the algorithm

We shall follow the key steps described in (Gronchi, 2002) to compute the solutions of the polynomial system (5); however we shall present some important improvements to that technique, allowing to reduce the computing time. These steps are

1. use the resultant theory to eliminate one variable;

2. compute the coefficients of the resultant polynomial (or of one factor of its) using an evaluation–interpolation method by the Fast Fourier Transform applied to the coefficients of the matrix defining the resultant (or defining its factor).

In the following we shall describe the algorithm in details.

4.1. Elimination of the variable s

From the *algebraic theory of elimination* (Cox et al., 1992) we know that $f(s,t)$ and $g(s,t)$ have a common factor (as polynomials in the variable s) if and only if the resultant $Res(t) = \mathtt{Res}(f(s,t), g(s,t), s)$ of f and g with respect to s is zero.

The resultant is given by the determinant of the Sylvester matrix

$$\mathtt{S}(t) = \begin{pmatrix} f_4 & 0 & g_2 & 0 & 0 & 0 \\ f_3 & f_4 & g_1 & g_2 & 0 & 0 \\ 0 & f_3 & g_0 & g_1 & g_2 & 0 \\ f_1 & 0 & 0 & g_0 & g_1 & g_2 \\ f_0 & f_1 & 0 & 0 & g_0 & g_1 \\ 0 & f_0 & 0 & 0 & 0 & g_0 \end{pmatrix},$$

that is

$$\begin{aligned} Res(t) &= -g_0 g_1^3 f_1 f_4 + 3\, g_0^2 g_1 g_2 f_1 f_4 + g_0 g_1^2 g_2 f_1 f_3 - g_1^3 g_2 f_0 f_3 - \\ &- g_1 g_2^3 f_0 f_1 + 3\, g_0 g_1 g_2^2 f_0 f_3 - g_0^3 g_1 f_3 f_4 - 4\, g_0 g_1^2 g_2 f_0 f_4 + 2\, g_0^2 g_2^2 f_0 f_4 + \\ &+ g_2^4 f_0^2 + g_0^4 f_4^2 + g_1^4 f_0 f_4 + g_0^3 g_2 f_3^2 - 2\, g_0^2 g_2^2 f_1 f_3 + g_0 g_2^3 f_1^2\,; \end{aligned}$$

and it is generically a 20-th degree polynomial in the variable t.

4.2. Factorization of the resultant

In a previous remark we have already observed that we know four solutions of (3) and then of (5): we want to use the basic properties of the determinants to extract a factor of degree 4 from the resultant.

Let $\alpha_E = \frac{E-1}{E+1}$. We note that

$$g_1(t) = \Big[t^2(e-1) - (e+1)\Big]\, \tilde{g}_1(t) \qquad (6)$$

$$g_2(t) + \alpha_E\, g_0(t) = \Big[t^2(e-1) - (e+1)\Big]\, \tilde{g}_{20}(t) \qquad (7)$$

$$f_3(t) + \alpha_E\, f_1(t) = \Big[t^2(e-1) - (e+1)\Big]\, \tilde{f}_{31}(t) \qquad (8)$$

where

$$\begin{aligned}\tilde{g}_1(t) &= 2P\left[N(e-1)t^2 - 2Lt + N(e+1)\right] ; \\ \tilde{g}_{20}(t) &= P\,(\alpha_E - 1)\left[M(e-1)\,t^2 - 2K\,t + M(e+1)\right] ; \\ \tilde{f}_{31}(t) &= -2EP(1+\alpha_E) .\end{aligned}$$

The resultant is equal to the determinant of the matrix

$$\tilde{\mathtt{S}}(t) = \begin{pmatrix} f_4 & 0 & g_2 & 0 & 0 & 0 \\ f_3 & f_4 & g_1 & g_2 & 0 & 0 \\ 0 & f_3 + \alpha_E f_1 & g_0 + g_2/\alpha_E & g_1 & g_2 + \alpha_E g_0 & \alpha_E g_1 \\ f_1 + f_3/\alpha_E & 0 & g_1/\alpha_E & g_0 + g_2/\alpha_E & g_1 & g_2 + \alpha_E g_0 \\ f_0 & f_1 & 0 & 0 & g_0 & g_1 \\ 0 & f_0 & 0 & 0 & 0 & g_0 \end{pmatrix}$$

obtained performing the following operations on the rows of $\mathtt{S}(t)$:

1. add to the 3rd row $1/\alpha_E$ times the 1st row and α_E times the 5th row;

2. add to the 4th row $1/\alpha_E$ times the 2nd row and α_E times the 6th row .

Using relations (6),(7),(8) and the basic properties of determinants we can write

$$Res(t) = det(\tilde{\mathtt{S}}(t)) = \left[t^2(e-1) - (e+1)\right]^2 det(\hat{\mathtt{S}}(t)) ,$$

with

$$\hat{\mathtt{S}}(t) = \begin{pmatrix} f_4 & 0 & g_2 & 0 & 0 & 0 \\ f_3 & f_4 & g_1 & g_2 & 0 & 0 \\ 0 & \tilde{f}_{31} & \tilde{g}_{20}/\alpha_E & \tilde{g}_1 & \tilde{g}_{20} & \alpha_E\,\tilde{g}_1 \\ \tilde{f}_{31}/\alpha_E & 0 & \tilde{g}_1/\alpha_E & \tilde{g}_{20}/\alpha_E & \tilde{g}_1 & \tilde{g}_{20} \\ f_0 & f_1 & 0 & 0 & g_0 & g_1 \\ 0 & f_0 & 0 & 0 & 0 & g_0 \end{pmatrix} ;$$

As the resultant $Res(t)$ is divisible by the factor $\left[t^2(e-1) - (e+1)\right]^2$ we can take into account the 16^{th} degree polynomial defined by

$$r(t) = det(\hat{\mathtt{S}}(t)) = \frac{Res(t)}{\left[t^2(e-1) - (e+1)\right]^2} .$$

REMARK: The factor $t^2(e-1) - (e+1)$ (for $e \neq 1$) has the roots $t = \pm\sqrt{\frac{e+1}{e-1}}$: these are purely imaginary if $e < 1$, while if $e > 1$ they correspond to the angular values of the asymptotes of the hyperbolic orbit. In any case these roots of the resultant have to be discarded. The term $t^2(e-1) - (e+1)$ corresponds to $1 + e\cos(v)$ in (3).

REMARK: The matrix $\hat{\mathtt{S}}(t)$ is not defined for $\alpha_E = 0$, and this singularity is not present in the original Sylvester matrix $\mathtt{S}(t)$. This can be explained as a wrong choice of the coordinate change in (4) that prevents us to find solutions at infinity and can be removed using the formulas described in Section 5.

REMARK: Applying (4) to system (3) with $E = 1$, the first equation in (5) has a smaller degree as a function of s than in the general case (see Appendix, Subsection 10.3): for this reason the determinant of the matrix $\mathtt{S}(t)$ becomes a multiple of the resultant $Res(t)$ of the two polynomials of the system with respect to s, in fact the Sylvester matrix of the system has in this case a smaller size (it is a 5×5 matrix). On the other hand if $e = 1$ the second equation in (5) has a smaller degree as a function of t, but the degree of the polynomials in the variable s is not smaller, so that the resultant $Res(t)$ can be computed as the determinant of $\mathtt{S}(t)$.

4.3. Computation of the coefficients of $r(t)$

We use the Fast Fourier Transform (FFT) to compute the coefficients of the polynomial $r(t) = det(\hat{\mathtt{S}}(t))$. The algorithms for the Discrete Fourier Transform (DFT) and the Inverse Discrete Fourier Transform (IDFT), that are respectively the FFT methods to perform evaluation and interpolation, are particularly efficient when working with a number of evaluations that is a power of 2. Unfortunately $r(t)$ has 17 coefficients ($= 2^4 + 1$).

We use the following strategy to work with a lower degree polynomial and use only 2^4 evaluations: we observe that we can write

$$r(t) = r_0 + t\,\tilde{r}(t) \tag{9}$$

where

$$\tilde{r}(t) = \sum_{j=0}^{15} r_{j+1}\, t^j \qquad \text{and} \qquad r_0 = det(\hat{\mathtt{S}}(0))\,.$$

We apply the evaluation–interpolation method to the 15^{th} degree polynomial $\tilde{r}(t)$ (with 2^4 coefficients) whose evaluations in the 16^{th} roots of unity are given by

$$\tilde{r}(e^{-2\pi i \frac{k}{16}}) = \frac{r(e^{-2\pi i \frac{k}{16}}) - r_0}{e^{-2\pi i \frac{k}{16}}}\,, \qquad k = 0 \ldots 15\ . \tag{10}$$

Thus we compute the coefficients of r by interpolating the values of $\tilde{r}$.

4.4. Steps of the algorithm

In this paragraph we explain the main steps of our method.

1. evaluate the polynomials $f_0, f_1, f_3, g_0, g_1, g_2, \tilde{g}_1, \tilde{g}_{20}$ [3] in $\hat{\mathtt{S}}(t)$ at $t = 0$ and at all the 16^{th} roots of unity

$$\omega_k = e^{-2\pi i \frac{k}{16}} \qquad k = 0, .., 15$$

   by the DFT algorithm ;

2. compute the determinant of the 16 matrixes $\hat{\mathtt{S}}(\omega_k)$, with $k = 0 \ldots 15$; each of them is evaluated at a different point of the complex plane. If a square matrix has its coefficients depending on a variable t, then the evaluation at a point $\overline{t}$ of the determinant of this matrix is equal to the determinant of the matrix whose coefficients are evaluated at $\overline{t}$. Thus we obtain the evaluation of $r(\omega_k)$, for $k = 0 \ldots 15$;

3. use (10) to compute $\tilde{r}(\omega_k)$ for $k = 0 \ldots 15$;

4. apply the IDFT algorithm to obtain the coefficients of $\tilde{r}(t)$ from its 16 evaluations ;

5. compute the coefficients of $r(t)$ using relation (9) ;

6. compute the real roots of $r(t)$. For this point we use the algorithm described in (Bini, 1997), based on simultaneous iterations ;

7. given a solution $\overline{t} \in \mathbb{R}$ of $r(t) = 0$, search for one or more values $\overline{s} \in \mathbb{R}$ for which $(\overline{t}, \overline{s})$ is a solution of (5) ; [4]

8. detect the type of singularity, i.e. classify the critical points in minimum, maximum or saddle points.

Note that even if the polynomial $r(t)$ can be written in a short form, its coefficients hide very long expressions, functions of the orbital elements. An advantage of the resultant method is that it allows to evaluate directly the coefficients of the matrix $\hat{\mathtt{S}}$, that are lower degree polynomials and have shorter expressions.

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[3] Note that $\tilde{f}_{31}$ is constant.

[4] This step is quite delicate, we have to deal with the following cases:

(i) for a real root $\overline{t}$ there are more than one real value $\overline{s}$ such that the pair $(\overline{t}, \overline{s})$ satisfies (5), indeed up to four values (see (Gronchi, 2002) for an example);

(ii) for a real root $\overline{t}$ there is a value $\overline{s} \in \mathbb{C} \setminus \mathbb{R}$, see Appendix, Subsection 10.4 for a simple example with low degree polynomials.

5. Shifts along the bounded orbits

In the case of bounded orbits (circles/ellipses) the variable change (4) does not allow to find the angular value π for V or v. If we know that $V^* + \pi$ and $v^* + \pi$ are not components of a critical point we can send one or both these values to infinity by composing (4) with an angular shift. We introduce the general variable change

$$\begin{cases} \Xi = V - \alpha \\ \xi = v - \beta \end{cases}$$

where Ξ, ξ are the new angular variables and α, β are constant angles.

By the usual trigonometric addition formulas applied to equation (1) we define the squared distance in terms of the unknowns (Ξ, ξ):

$$\delta^2(\Xi, \xi) = \langle \mathcal{X}_1 - \mathcal{X}_2, \mathcal{X}_1 - \mathcal{X}_2 \rangle$$

where

$$\mathcal{X}_1 = R\,[\mathcal{A} \cos\Xi + \mathcal{B} \sin\Xi] \ ; \qquad \mathcal{X}_2 = r\,[\mathfrak{a} \cos\xi + \mathfrak{b} \sin\xi] \ ;$$

and

$$\begin{array}{ll} \mathcal{A} = \mathcal{P} \, \cos\alpha + \mathcal{Q} \, \sin\alpha \, ; & \mathcal{B} = -\mathcal{P} \, \sin\alpha + \mathcal{Q} \, \cos\alpha \, ; \\ \mathfrak{a} = \mathfrak{p} \, \cos\beta + \mathfrak{q} \, \sin\beta \, ; & \mathfrak{b} = -\mathfrak{p} \, \sin\beta + \mathfrak{q} \, \cos\beta \, ; \end{array}$$

with components defined by

$$\begin{array}{ll} \mathcal{A} = (A_x \, , A_y \, , A_z) \ ; & \mathcal{B} = (B_x \, , B_y \, , B_z) \ ; \\ \mathfrak{a} = (a_x \, , a_y \, , a_z) \ ; & \mathfrak{b} = (b_x \, , b_y \, , b_z) \ . \end{array}$$

The system defining the critical points is

$$\nabla_{\Xi,\xi} \delta^2(\Xi, \xi) = 0 \ , \tag{11}$$

and the components of the gradient are

$$\frac{\partial \delta^2}{\partial \Xi} = 2 \langle \mathcal{X}_1 - \mathcal{X}_2, \frac{\partial}{\partial \Xi}(\mathcal{X}_1 - \mathcal{X}_2) \rangle \, ; \qquad \frac{\partial \delta^2}{\partial \xi} = 2 \langle \mathcal{X}_1 - \mathcal{X}_2, \frac{\partial}{\partial \xi}(\mathcal{X}_1 - \mathcal{X}_2) \rangle \, ;$$

where

$$\begin{aligned} \frac{\partial}{\partial \Xi}(\mathcal{X}_1 - \mathcal{X}_2) &= \frac{P}{(1 + E \cos V)^2} \, [\mathcal{B} \, \cos\Xi - \mathcal{A} \, \sin\Xi + \mathcal{Q} \, E] \ ; \\ \frac{\partial}{\partial \xi}(\mathcal{X}_1 - \mathcal{X}_2) &= \frac{p}{(1 + E \cos v)^2} \, [\mathfrak{b} \, \cos\xi - \mathfrak{a} \, \sin\xi + \mathfrak{q} \, e] \ . \end{aligned}$$

Remark: The following relations hold:

$$\|\mathcal{A}\| = \|\mathcal{B}\| = \|\mathfrak{a}\| = \|\mathfrak{b}\| = 1 \, ; \qquad \langle \mathcal{A}, \mathcal{B} \rangle = \langle \mathfrak{a}, \mathfrak{b} \rangle = 0 \, ;$$

$$\mathcal{A}|_{\alpha=0} = \mathcal{P}\,; \quad \mathcal{B}|_{\alpha=0} = \mathcal{Q}\,; \quad \mathfrak{a}|_{\beta=0} = \mathfrak{p}\,; \quad \mathfrak{b}|_{\beta=0} = \mathfrak{q}\ .$$

From (11) we obtain

$$\left\{ \begin{array}{l} p\,[1+E\cos(\Xi+\alpha)]\,\langle \mathcal{P}\sin(\Xi+\alpha) - \mathcal{Q}\,[E+\cos(\Xi+\alpha)]\,, \mathfrak{a}\cos\xi + \mathfrak{b}\sin\xi\rangle + \\ +EP\sin(\Xi+\alpha)\,[1+e\cos(\xi+\beta)] = 0\,; \\ \\ P\,[1+e\cos(\xi+\beta)]\,\langle \mathfrak{p}\sin(\xi+\beta) - \mathfrak{q}\,[e+\cos(\xi+\beta)]\,, \mathcal{A}\cos\Xi + \mathcal{B}\sin\Xi\rangle + \\ +ep\sin(\xi+\beta)\,[1+E\cos(\Xi+\alpha)] = 0\ . \end{array} \right. \tag{12}$$

REMARK: The values of the pairs (Ξ, ξ) such that $1 + E\cos(\Xi + \alpha) = 1 + e\cos(\xi + \beta) = 0$ are always solutions of system (12). Their explicit values are

$$\Xi = \pm(\arccos(-1/E) - \alpha)\,; \qquad \xi = \pm(\arccos(-1/e) - \beta)\ .$$

Using the variable change

$$\left\{ \begin{array}{ll} z & = \tan(\Xi/2) \\ w & = \tan(\xi/2) \end{array} \right. \tag{13}$$

we can transform system (12) into a polynomial system in the variables z, w and we can generalize the procedure described in Sections 3, 4 to find its solutions, see Subsections 10.1, 10.2 in the Appendix for the details.

6. Size of the solutions along the unbounded orbits

As we have seen in the previous sections, there are natural bounds to the true anomalies V, v of the critical points of d^2:

$$1 + E\cos V > 0\,; \qquad 1 + e\cos v > 0\ .$$

If we consider two unbounded orbits then it is not possible to set additional bounds to the components of the critical points: think about the example of two overlapping hyperbolic orbits.

On the other hand, if the unbounded orbit is only one, we can set a more restrictive bound on the component of the critical points along this orbit.

It is useful to remind a geometric interpretation of the stationary points of the squared distance d^2 between two any smooth curves $\gamma_1(V), \gamma_2(v)$ in $\mathbb{R}^3$:

LEMMA 1. *If $(\overline{V}, \overline{v})$ is a critical point of d^2, and $P_1 = \gamma_1(\overline{V}), P_2 = \gamma_2(\overline{v})$ are the Cartesian coordinates in $\mathbb{R}^3$ that correspond to it on the two curves, then the straight line joining P_1 and P_2 is orthogonal to both the tangent lines to γ_1 and γ_2 in P_1, P_2* (see Figure 1).

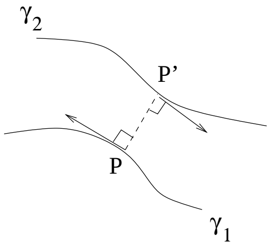

Figure 1 The geometry of the stationarity condition for the distance function between two curves: the line joining the Cartesian coordinates of the critical points must be orthogonal to both tangent vectors.

Proof. We have $d^2(V,v) = \langle \gamma_1(V) - \gamma_2(v), \gamma_1(V) - \gamma_2(v) \rangle$, so that

$$\begin{aligned}\frac{\partial d^2}{\partial V}(\overline{V},\overline{v}) &= 2\left\langle \frac{d\gamma_1}{dV}(\overline{V}), P_1 - P_2 \right\rangle = 0 \\ \frac{\partial d^2}{\partial v}(\overline{V},\overline{v}) &= 2\left\langle \frac{d\gamma_2}{dv}(\overline{v}), P_1 - P_2 \right\rangle = 0\end{aligned}$$

□

Let us consider the case of a planet and a non-periodic comet. The parametric equation of the orbit of the comet γ_2 in terms of the true anomaly v, in a reference frame with the x axis pointing towards the pericenter of γ_2 and the y axis lying on the plane of this orbit, is given by

$$\gamma_2 \equiv \left(\frac{p\cos v}{1+e\cos v}, \frac{p\sin v}{1+e\cos v}, 0 \right)$$

where $p = a(1-e^2)$ is the conic parameter. For each point $P \in \gamma_2$, corresponding to a value v, we write the tangent vector $\tau(P)$ to γ_2 in P as

$$\tau(P) = \frac{1}{\sqrt{1+2e\cos v + e^2}} \left(-\sin v, \cos v + e, 0 \right) \ .$$

The plane π orthogonal to $\tau(P)$ and passing through P is given by

$$-\sin v\, x + (\cos v + e)\, y - F(v) = 0\,,$$

where

$$F(v) = \frac{p\, e\, \sin v}{1+e\cos v} \ .$$

The squared distance from π to the origin O is the minimum of the function

$$\delta^2(x) = x^2 \left[1 + \frac{\sin^2 v}{(\cos v + e)^2} \right] + 2\, \frac{x\, \sin v\, F(v)}{(\cos v + e)^2} + \frac{F^2(v)}{(\cos v + e)^2}\,,$$

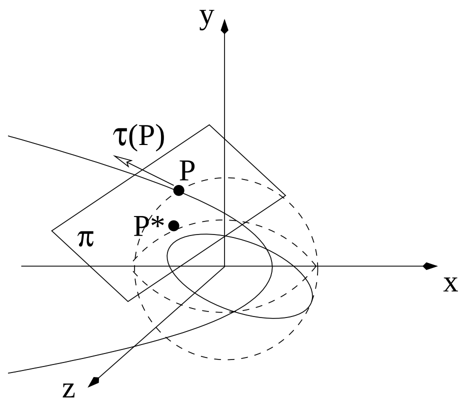

Figure 2 Geometrical sketch of the estimate (14), that gives a bound for the parameter of a critical point along the comet orbit (parabolic or hyperbolic). The point P^* corresponds to the tangency of the plane π to the sphere with radius R_a, the apocenter distance of the orbit of the planet.

that is attained in

$$x^* = -\frac{\sin v\, F(v)}{1 + 2\, e \cos v + e^2} \ .$$

We obtain

$$\delta^2(x^*) = \frac{F^2(v)}{1 + 2\, e \cos v + e^2} = \frac{p^2\, e^2 \sin^2 v}{(1 + e\cos v)^2\, (1 + 2\, e \cos v + e^2)} \ .$$

Let R_a be the apocenter distance of the planet orbit γ_1 and let us set $\xi = \cos v$, that implies $\xi \in]-1/e, 1]$. Thus Lemma (1) implies that the ξ component of a critical point has to fulfill the relation

$$R_a^2 (1 + e\xi)^2\, (1 + 2\, e\, \xi + e^2) \geq e^2 p^2 (1 - \xi^2) \ . \tag{14}$$

Remark: For $\xi = 1$ relation (14) trivially holds. We can then assume in the following that $\xi \in]-1/e, 1[$.

Let us define the functions

$$h(\xi) = \frac{R_a^2 (1 + e\xi)^2}{1 - \xi^2} \ ; \qquad k(\xi) = \frac{p^2 e^2}{1 + 2e\xi + e^2} \ ;$$

then relation (14) on the interval $]-1/e, 1[$ can be written as $h(\xi) \geq k(\xi)$.

A simple computation of the derivatives of h, k shows that $h(\xi)$ is strictly increasing in the interval considered, and $k(\xi)$ is strictly decreasing; furthermore

$$h(-1/e) = 0\,; \qquad \lim_{\xi\to -1/e^+} k(\xi) = \begin{cases} \dfrac{p^2e^2}{e^2-1} > 0 & \text{if } e>1 \\ +\infty & \text{if } e=1 \end{cases}\;;$$

$$h(0) = R_a^2\,; \qquad k(0) = \frac{p^2e^2}{1+e^2}\,;$$

$$\lim_{\xi\to 1^-} h(\xi) = +\infty\,; \qquad k(1) = \frac{p^2e^2}{(1+e)^2}\;.$$

From these considerations, using the monotonicity properties of h, k we know that there is always only one point $\xi^* \in]-1/e, 1[$ such that $h(\xi^*) = k(\xi^*)$; furthermore condition (14) gives $\xi \geq \xi^*$, that is

$$-\arccos(\xi^*) \leq v \leq \arccos(\xi^*). \tag{15}$$

Then the maximum value of the distance from the focus r_{max} for the Cartesian components of a critical point along the orbit of the comet is given by

$$r_{max} = \frac{p}{1+e\xi^*}\;.$$

The point ξ^* is one of the roots of the third degree equation

$$R_a^2\,(1+2e\xi+e^2)(1+e\xi)^2 = p^2e^2\,(1-\xi^2)\;.$$

Using the monotonicity properties of h, k we can give a bound to the size of ξ^*: we observe that

1. if $h(0) < k(0)$ then $0 < \xi^* < \xi_{max}$;
2. if $h(0) > k(0)$ then $\xi_{min} < \xi^* < 0$;

where ξ_{max} is the positive solution of the second degree equation given by $h(\xi) = k(0)$, while ξ_{min} is the solution of $k(\xi) = h(0)$, so that $|\xi^*| \leq \max\{|\xi_{min}|, \xi_{max}\}$.

REMARK: The estimate (15) is optimal, in fact it can be attained if the apocenter of the bounded orbit corresponds to the point marked with P^* in Figure (2). It is even possible for the distance to vanish for $v = v^* = \pm\arccos(\xi^*)$ if the apocenter of the planet orbit $\gamma_1(\pi)$ and $\gamma_2(v^*)$ coincide with P^*.

REMARK: If $e = 1$ relation (14) becomes

$$2R_a^2(1+\xi)^2 \geq p^2(1-\xi)\,,$$

and we obtain a simple expression for ξ^*:

$$\xi^* = \frac{1}{4R_a^2}\left[-(p^2+4R_a^2) + p\sqrt{p^2+16R_a^2}\right]\;.$$

REMARK: If the apocenter distance R_a is $\leq pe/\sqrt{1+e^2}$, then $\xi^* \geq 0$, $v \in [-\pi/2, \pi/2]$ and $r_{max} = p$.

7. Infinitely many critical points

In the case of two concentric coplanar circles or two overlapping conics we have trivially an infinite number of critical points of d^2. We shall show that these cases are the only with this property.

PROPOSITION 1. *Let us consider two Keplerian orbits with a common focus. If there are infinitely many critical points of the squared distance function d^2 between these orbits, then either the two orbits are concentric coplanar circles or they are two overlapping conics.*

Proof. If there are infinitely many real solutions of system (24) [5], then the two polynomials have a common factor $\mathsf{h}_{\alpha,\beta}(z,w)$ (with total degree less or equal to 4) with a *continuum* of real roots, that correspond to critical points of d^2.

The singular points of the polynomial $\mathsf{h}_{\alpha,\beta}(z,w)$ are isolated, hence there exists an open set in the plane (z,w) containing regular points of $\mathsf{h}_{\alpha,\beta}$, such that $\mathsf{h}_{\alpha,\beta}(z,w) = 0$. Then we can define a regular parametric curve $\Gamma :]-1,1[\rightarrow \mathbb{R}^2$, with parameter σ, such that $\Gamma(\sigma)$ is a critical point of d^2 for each $\sigma \in]-1,1[$.

The value of d^2 along the curve Γ is a constant ρ: this can be easily checked by computing the derivative of $d^2(\Gamma(\sigma))$ with the *chain rule*.

Let us take into account the first orbit γ_1 and draw the smooth surface Σ composed by the union of the circles with radius ρ centered in the points of γ_1 and orthogonal to γ_1 at these points. Consider now a plane passing through a focus of γ_1 (the common focus) and not coinciding with the first orbit plane: we shall show that no section cut by this plane on the surface Σ can be an arc of conic, not even locally.

We begin with the simplest case: γ_1 is a circular orbit with radius R. Assuming that γ_1 is on the plane (X_1, X_2), then the surface Σ (which is the ordinary torus) has parametric equations

$$\begin{cases} X_1 = \cos V(R + \rho\cos\phi) \\ X_2 = \sin V(R + \rho\cos\phi) \\ X_3 = \rho\sin\phi \end{cases}$$

with parameters V, ϕ.

[5] We are considering the general polynomial formulation with the angular shifts given in the Appendix.

The plane π passing through the focus O, where the second orbit lies, is defined by

$$AX_1 + BX_2 + CX_3 = 0$$

for some constants $A, B, C \in \mathbb{R}$. Assuming that this plane is not orthogonal to the z axis gives us the relation $A^2 + B^2 > 0$.

We select two vectors $\hat{e}_1, \hat{e}_2 \in \mathbb{R}^3$ that generate a Cartesian reference frame on the plane π. Choosing $\hat{e}_1$ on the line where the two orbital planes intersect we have

$$\hat{e}_1 = (-B, A, 0)\,; \qquad \hat{e}_2 = \frac{1}{\sqrt{1+C^2}}(-CA, -CB, 1)\,;$$

with $A^2 + B^2 = 1$.

Using Cartesian coordinates (ξ, η) on the plane π, we write the vector equation

$$\xi\,\hat{e}_1 + \eta\,\hat{e}_2 = (X_1, X_2, X_3)$$

or, more explicitely

$$\begin{cases} -\xi\,B - \eta\,\dfrac{CA}{\sqrt{1+C^2}} = \cos V(R + \rho\cos\phi) \\[2ex] \xi\,A - \eta\,\dfrac{CB}{\sqrt{1+C^2}} = \sin V(R + \rho\cos\phi) \\[2ex] \dfrac{\eta}{\sqrt{1+C^2}} = \rho\sin\phi \end{cases} \tag{16}$$

that are three equations in the four unknowns ξ, η, V, ϕ.

We want to perform an elimination of variables and write only one equation relating ξ and η. From the third equation in (16) we immediately obtain [6]

$$\sin\phi = \frac{\eta}{\rho\sqrt{1+C^2}}\,; \qquad \cos^2\phi = \frac{1}{\rho^2}\left[\rho^2 - \frac{\eta^2}{1+C^2}\right]\,; \tag{17}$$

hence we can write $\sin\phi, \cos\phi$ as functions of ξ, η. Squaring and summing the first two equations in (16) we have

$$\xi^2 + \frac{C^2\eta^2}{1+C^2} = (R + \rho\cos\phi)^2$$

and, by (17),

$$\xi^2 + \eta^2 - (R^2 + \rho^2) = \pm 2R\sqrt{\rho^2 - \frac{\eta^2}{1+C^2}}\,. \tag{18}$$

[6] $\rho > 0$ otherwise the two orbital planes would coincide.

The last equations can not represent an arc of a conic, not even locally, as can be easily seen by using polar coordinates (r, θ) defined by $\xi = r\cos(\theta), \eta = r\sin(\theta)$. In fact if it were, from the general equation of a conic in polar coordinates $r = p/(1 + e\cos(\theta))$, with eccentricity e and conic parameter p, we would have

$$\xi = \frac{p-r}{e}\,; \qquad \eta^2 = \frac{e^2r^2 - (p-r)^2}{e^2}\,; \tag{19}$$

thus, substituting in (18), we would obtain the relation

$$r^2 - C_1 = \pm C_2\sqrt{C_3 - e^2r^2 + (p-r)^2} \tag{20}$$

for positive constants C_1, C_2, C_3, that can not be true for each value of r in an open interval. [7]

Then we have to show that also the case of a circular arc (r = constant) is excluded. From (18) with constant $r = r_0$ we obtain

$$r_0^2 - (R^2 + \rho^2) = \pm 2R\sqrt{\rho^2 - \frac{r_0^2\sin^2\theta}{1 + C^2}}\,,$$

that can not hold for θ in an open interval.

We study the case of two coincident orbital planes by passing to the limit for $C \to +\infty$. Then (18) becomes

$$r^2 - (R^2 + \rho^2) = \pm 2R\rho\,,$$

that gives the radius of two circular orbits, coplanar with γ_1.

Now we shall consider the general case of a conic γ_1 with equation in polar coordinates (R, V)

$$R(V) = \frac{P}{1 + E\cos V}\,; \qquad P = Q(1 + E)\,;$$

where P is the conic parameter, Q the pericenter distance and the eccentricity E is assumed > 0.

The surface Σ is defined by

$$\begin{cases} X_1 = R(V)\cos V + \rho\cos[\alpha(V)]\cos\phi \\ X_2 = R(V)\sin V + \rho\sin[\alpha(V)]\cos\phi \\ X_3 = \rho\sin\phi \end{cases}$$

where

$$\begin{cases} \cos[\alpha(V)] = \dfrac{\cos V + E}{\sqrt{1 + 2E\cos V + E^2}} \\[2ex] \sin[\alpha(V)] = \dfrac{\sin V}{\sqrt{1 + 2E\cos V + E^2}} \end{cases}$$

[7] By squaring both sides of (20) we obtain a polynomial in the variable r.

Following the same steps of the previous case we obtain the system

$$\begin{cases} -\xi\, B - \eta\, \dfrac{CA}{\sqrt{1+C^2}} = R(V)\cos V + \rho\cos[\alpha(V)]\cos\phi \\ \\ \xi\, A - \eta\, \dfrac{CB}{\sqrt{1+C^2}} = R(V)\sin V + \rho\sin[\alpha(V)]\cos\phi \\ \\ \dfrac{\eta}{\sqrt{1+C^2}} = \rho\sin\phi \end{cases} \tag{21}$$

and, by squaring and summing the three equations in (21), we obtain

$$\xi^2 + \eta^2 = R^2(V) + \rho^2 + \frac{2P\rho\cos\phi}{1 + 2E\cos V + E^2} \tag{22}$$

where $\cos\phi$ is given in terms of η by formula (17).

Note that we have not eliminated the dependence on V. We can compute $\cos V$ as a function of ξ, η using the first equation in (21).

We complete the proof by contradiction: no arc of conic can satisfy equation (22), in fact if it were, using relations (19) we would obtain an equation $\mathfrak{e}(r) = 0$ in the variable r that have at most a discrete number of solutions. Actually, even if this equation is not as simple as (20), we can write it by performing on some powers of r a finite number of sums, multiplications by constant and root extractions. The left hand side $\mathfrak{e}(r)$ of such equation is an analytic function of r as a complex variable, except for at most a countable number of points.

As a result we obtain only constant solutions for r and, if r is constant, the second orbit must be circular. Then we can apply a reciprocity argument starting from this circular orbit and using the results previously shown to prove that also the first orbit should be circular, and that is a contradiction.

Also in this case we can deal with coincident orbital planes by passing to the limit for $C \to +\infty$. From (21) it follows that

$$\rho = 0 \quad \text{or} \quad \sin(\phi) = 0\ .$$

If $\rho \equiv 0$ we have two coincident conics, otherwise $\sin(\phi) = 0$, so that $\cos(\phi) = \pm 1$. We can exclude the last case again by contradiction: using an argument similar to the previous one, we obtain an equation in the variable r that can not be true for each value of r in an open interval. □

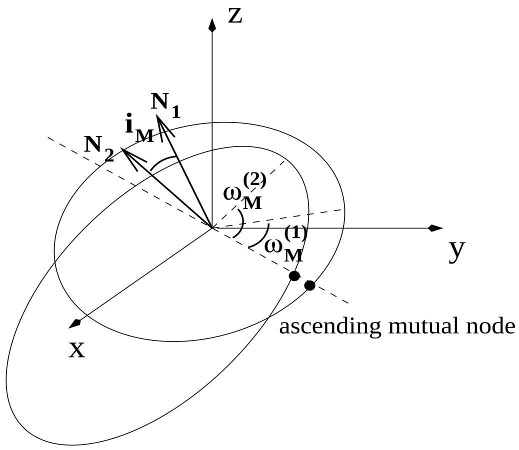

Figure 3 We draw some of the mutual elements for two orbits. Note the direction of the orientation vectors $\mathbf{N}_1, \mathbf{N}_2$, that defines the mutual inclination i_M and the mutual ascending node.

8. Numerical experiments and applications to Solar system orbits

8.1. Large scale experiments

To make a large number of numerical experiments with different orbital configurations we can take advantage of a set of elements depending only on the mutual position of the two orbits.

Given two Keplerian orbits with a common focus and nonzero mutual inclination, we define the *cometary mutual elements*

$$\mathcal{E}_M = \{Q, E, q, e, i_M, \omega_M^{(1)}, \omega_M^{(2)}\}$$

as follows: Q, E and q, e are the *pericenter distance* and the *eccentricity* of the two orbits, i_M is the *mutual inclination* between the two orbital planes and $\omega_M^{(1)}, \omega_M^{(2)}$ are the angles between the *ascending mutual node* of the second orbit with respect to the first orbit and the pericenters of the two orbits. [8]

The map

$$\Phi : (\mathcal{E}_1, \mathcal{E}_2) \to \mathcal{E}_M$$

[8] These elements are defined by assigning an orientation to both orbits, i.e. a normal vector $\mathbf{N}_i$ $(i = 1, 2)$ to each orbital plane. The mutual inclination is the angle between $\mathbf{N}_1$ and $\mathbf{N}_2$, while the ascending mutual node corresponds to the pair of points defined by the intersection of the two orbits with the mutual node line, that lies on the same side with respect to the origin as the wedge product $\mathbf{N}_1 \wedge \mathbf{N}_2$ of the two orientation vectors (see Figure 3).

from the ordinary cometary elements to the mutual elements, is *not* injective, actually there are infinitely many configurations that brings to the same mutual position of the two orbits. [9] We define an inverse of the map Φ by selecting a set of elements $(\mathcal{E}_1, \mathcal{E}_2)$ in each counter-image $\Phi^{-1}(\mathcal{E}_M)$:

$$\begin{aligned} \mathcal{E}_1 &= \{Q, E, i_1, \Omega_1, \omega_1\} = \{Q, E, 0, 0, \omega_M^{(1)}\}, \\ \mathcal{E}_2 &= \{q, e, i_2, \Omega_2, \omega_2\} = \{q, e, i_M, 0, \omega_M^{(2)}\}. \end{aligned} \tag{23}$$

Using the axial symmetry of conics we realize that the transformations

$$\begin{cases} \omega_M^{(1)} \to \pi - \omega_M^{(1)} \\ \omega_M^{(2)} \to \pi - \omega_M^{(2)} \end{cases} \qquad \begin{cases} \omega_M^{(1)} \to \pi + \omega_M^{(1)} \\ \omega_M^{(2)} \to \pi + \omega_M^{(2)} \end{cases} \qquad \begin{cases} \omega_M^{(1)} \to 2\pi - \omega_M^{(1)} \\ \omega_M^{(2)} \to 2\pi - \omega_M^{(2)} \end{cases}$$

give rise to the same critical values of the distance. Therefore we only need to take into account the values of $i_M, \omega_M^{(1)}, \omega_M^{(2)}$ in the following ranges:

$$i_M \in]0, \pi[\,; \qquad \omega_M^{(1)} \in [0, \pi/2[\,; \qquad \omega_M^{(2)} \in [0, 2\pi[\;.$$

Using mutual cometary elements and the map (23) we have been able to perform a large number of numerical experiments with significantly different orbital configurations, avoiding to compute the critical points of d^2 for configurations that give the same critical values. We have also identified some cases with a high number of critical points.

In Figure 4 we show the level lines of the squared distance d^2 for an example with 10 critical points: note that one orbit is circular. The values of the critical points, the corresponding values of d and the type of singularity are displayed in Table I. This example could appear artificial, but we can find similar cases even among the Near Earth Asteroids: see for example the 10 critical points for the asteroid 2004 *LG* with respect to the Earth orbit on the NEODyS website.[10]

In Figure 5 we show the level lines of d^2 for an example with 12 critical points, the maximal number of points that we have found within these experiments: the values of the critical points, the corresponding values of d and the type of singularity are displayed in Table II.

In Figures 6, 7 we present two cases with an elliptic and a hyperbolic orbit: we obtain in both cases 6 critical points, that is the largest

[9] e.g. we can rotate by the same angle both orbits around an axis passing through the common focus without changing their mutual position.

[10] The *Near Earth Asteroids Dynamic Site* at the University of Pisa: web address `http://newton.dm.unipi.it/neodys`

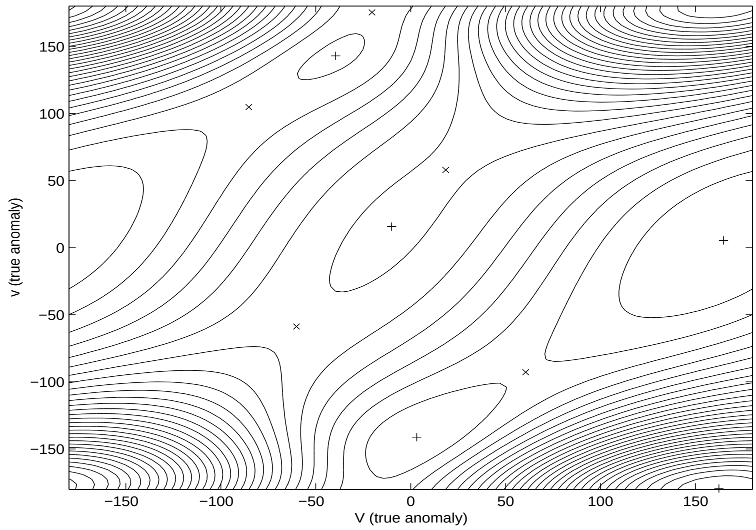

Figure 4 Level curves of the squared distance for an example with 10 critical points: the local extrema are marked with a *plus* while saddle points are marked with a *cross*.

Table I An example with 10 critical points: in the table we write the corresponding values of the true anomalies (in degrees), the values of the distance d and the type of singularity: note that one of the two conics is a circle (see Table III).

V	v	distance	type
164.70127	5.40234	0.51940	MINIMUM
3.18796	-141.16197	0.75687	MINIMUM
-39.54070	142.93388	0.86458	MINIMUM
60.52617	-92.83135	0.90461	SADDLE
-20.41060	175.23045	0.92827	SADDLE
-85.28388	104.70790	0.93224	SADDLE
-60.11674	-58.72173	1.44587	SADDLE
18.44302	57.90583	1.47347	SADDLE
-10.06618	15.74301	1.48171	MAXIMUM
162.29077	-179.41542	2.91897	MAXIMUM

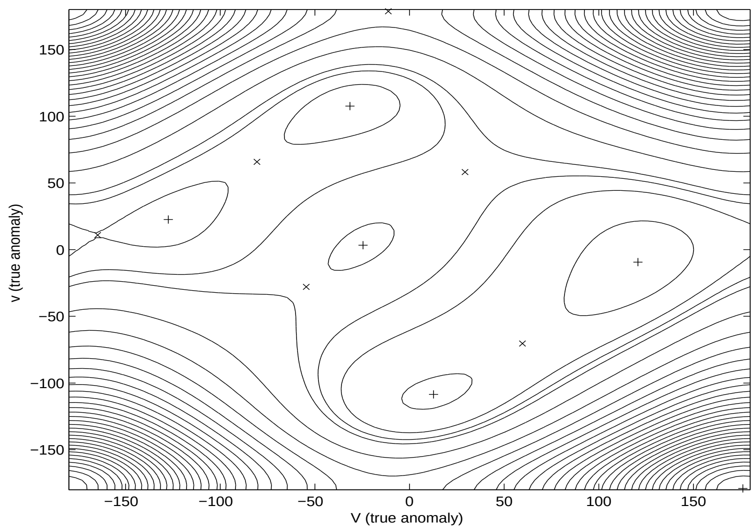

Figure 5 Level curves of the squared distance for an example with 12 critical points.

Table II An example with 12 critical points.

V	v	distance	type
120.68556	-9.33288	0.83357	MINIMUM
12.71196	-108.56712	0.86807	MINIMUM
59.69387	-70.40595	0.89802	SADDLE
-31.44700	107.56234	0.94700	MINIMUM
-127.41750	22.52194	0.95415	MINIMUM
-164.74517	10.89872	0.96957	SADDLE
-80.56016	65.78350	0.97555	SADDLE
29.32904	58.13570	1.03159	SADDLE
-54.54877	-27.88305	1.04803	SADDLE
-24.51761	3.34997	1.05248	MAXIMUM
-11.19971	178.71433	1.35307	SADDLE
176.16645	-179.01403	3.34646	MAXIMUM

number that we have found with one unbounded orbit. In the second case we have 3 minimum, no maximum and 3 saddle points: this is possible only with unbounded orbits because in this case the existence of a maximum point is no more granted (the domain $\mathbb{R} \times S^1$ is no more compact). The values of the critical points, the corresponding values of d and the type of singularity are given in Tables IV,V.

The mutual elements used for these 4 examples are given in Table III.

Table III Mutual elements for the examples given in this section, with the number of the figures they are referring to.

Figure number	Q	e_1	q	e_2	i_M	$\omega_M^{(1)}$	$\omega_M^{(2)}$
4	1.0	0.0	0.48	0.6	60.0°	16.0°	176.0°
5	0.585	0.415	0.462	0.615	80.0°	8.0°	176.0°
6	1.0	0.6	1.2	1.1	40.0°	73.0°	69.0°
7	1.0	0.5	1.2	1.1	66.0°	4.0°	136.0°

Table IV Critical points for the example in Figure 6.

V	v	distance	type
-69.49877	-58.67705	0.34619	MINIMUM
76.74888	69.25935	0.81742	MINIMUM
46.83819	44.61670	0.83243	SADDLE
-169.88880	62.56604	4.94731	SADDLE
169.88879	-56.53012	5.00016	SADDLE
176.02598	-20.46019	5.00725	MAXIMUM

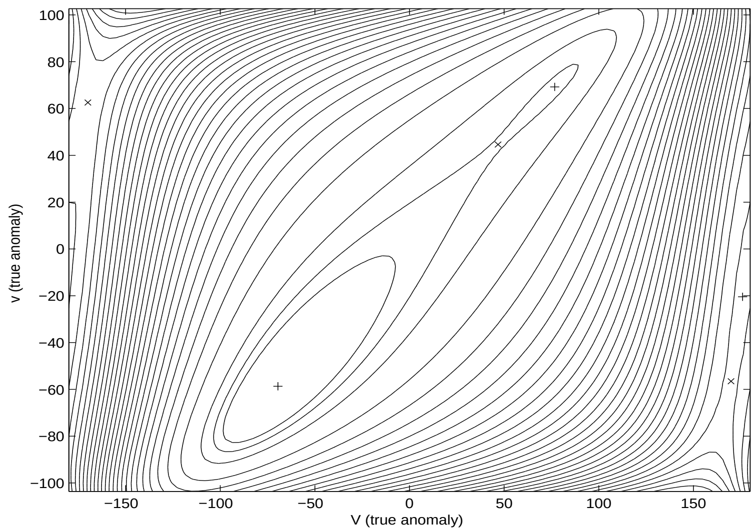

Figure 6 Level curves of the squared distance for an elliptic and a hyperbolic orbit: in this case we find 2 minimum and 1 maximum points (see Table IV).

Table V Critical points for the example in Figure 7.

V	v	distance	type
-160.6036221	66.6649070	1.44214	MINIMUM
52.8597535	-53.9730298	1.48730	MINIMUM
138.6616780	32.7954913	1.50853	MINIMUM
160.4380015	50.0738056	1.51541	SADDLE
102.1493828	-8.3520246	1.52564	SADDLE
-73.5585717	7.6851159	2.18797	SADDLE

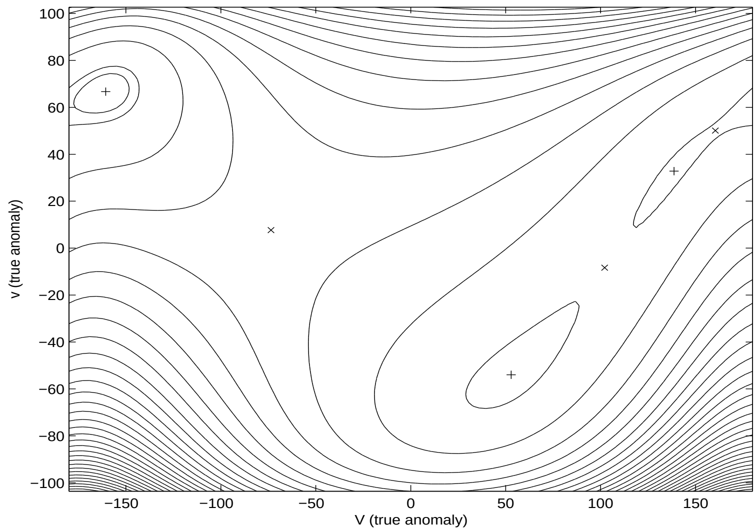

Figure 7 Level curves of the squared distance for an elliptic and a hyperbolic orbit: in this case we find 3 minimum and no maximum points (see Table V). This is possible only with unbounded orbits.

We also present in Table VI the results of the computation of the MOID between asteroid orbits from the catalog of the ASTDyS website[11] with absolute magnitude ≤ 8 and semimajor axis ≤ 10 AU: we write in this table all the cases with MOID ≤ 0.001 AU.

9. Conclusions and future work

We have introduced an algebraic method to compute the critical points of the distance function between two orbits: this algorithm can be

[11] The *Asteroids Dynamic Site* at the University of Pisa: web address `http://hamilton.dm.unipi.it/astdys`

Table VI The pairs of numbered asteroids with semimajor axis ≤ 10 and absolute magnitude $H \leq 8$ such that the MOID of their orbits is ≤ 0.001 AU.

1st asteroid number	H	2nd asteroid number	H	MOID (AU)
10	5.360	48	6.920	0.0005346
7	5.460	20	6.400	0.0001038
7	5.460	115	7.470	0.0002608
6	5.660	43	7.600	0.0004686
532	5.880	511	6.170	0.0003827
16	5.910	324	6.850	0.0006213
39	6.050	61	7.510	0.0002160
9	6.210	804	7.720	0.0000740
14	6.270	144	7.880	0.0006550
52	6.270	579	7.710	0.0004601
52	6.270	211	7.720	0.0006535
20	6.400	55	7.630	0.0000689
11	6.480	13	6.690	0.0004247
11	6.480	17	7.510	0.0002454
31	6.660	416	7.620	0.0006301
471	6.680	230	7.290	0.0001094
471	6.680	194	7.550	0.0006347
57	6.730	104	7.970	0.0005222
324	6.850	104	7.970	0.0003598
27	6.890	116	7.710	0.0009039
130	6.950	100	7.500	0.0004337
28	6.960	17	7.510	0.0003593
216	6.970	179	7.940	0.0000599
23	6.970	702	7.240	0.0005057
192	7.050	849	7.920	0.0001213
202	7.060	674	7.240	0.0006416
250	7.270	595	7.810	0.0009579
51	7.290	287	8.000	0.0002824
128	7.310	110	7.600	0.0001421
37	7.320	85	7.450	0.0008676
42	7.340	145	7.950	0.0006918
96	7.480	55	7.630	0.0004209
148	7.500	152	7.990	0.0002459
194	7.550	154	7.640	0.0001583
194	7.550	779	7.830	0.0009616
54	7.640	579	7.710	0.0002502
76	7.770	595	7.810	0.0001334
76	7.770	168	7.830	0.0005114
59	7.910	762	7.960	0.0008442
70	7.960	152	7.990	0.0009790

efficiently used to compute the MOID between two confocal orbits. We can use the information given by the MOID for different purposes, for example to measure the impact hazard of Near Earth Asteroids with the Earth. The speed and robustness of this algorithm is such to allow also large scale computations, to select in reasonable time pairs of asteroids suitable to be used for the problem of mass determination. In fact we can use this algorithm as a filter to select, among the orbits of all the asteroids, the ones with low MOID with respect to the orbits of big asteroids: we can propagate them forward in time and, if there is a close approach (possible only if the MOID is small), we can study their deflection.

10. Appendix

10.1. Algebraic formulation with the angular shifts

Using the variable change (13) and the relations

$$\begin{aligned} 1 + E\cos(\Xi+\alpha) &= \frac{1}{1+z^2}\left[(1-E\cos\alpha)z^2 - 2zE\sin\alpha + (1+E\cos\alpha)\right] ; \\ \sin(\Xi+\alpha) &= \frac{1}{1+z^2}\left[-z^2\sin\alpha + 2z\cos\alpha + \sin\alpha\right] ; \\ E + \cos(\Xi+\alpha) &= \frac{1}{1+z^2}\left[(E-\cos\alpha)z^2 - 2z\sin\alpha + (E+\cos\alpha)\right] ; \end{aligned}$$

$$\begin{aligned} 1 + e\cos(\xi+\beta) &= \frac{1}{1+w^2}\left[(1-e\cos\beta)w^2 - 2we\sin\beta + (1+e\cos\beta)\right] ; \\ \sin(\xi+\beta) &= \frac{1}{1+w^2}\left[-w^2\sin\beta + 2w\cos\beta + \sin\beta\right] ; \\ e + \cos(\xi+\beta) &= \frac{1}{1+w^2}\left[(e-\cos\beta)w^2 - 2w\sin\beta + (e+\cos\beta)\right] ; \end{aligned}$$

we transform the problem (12) into the polynomial system

$$\begin{cases} \mathsf{f}_{\alpha,\beta}(z,w) = \mathsf{f}_4(w)\,z^4 + \mathsf{f}_3(w)\,z^3 + \mathsf{f}_2(w)\,z^2 + \mathsf{f}_1(w)\,z + \mathsf{f}_0(w) = 0 \\ \mathsf{g}_{\alpha,\beta}(z,w) = \mathsf{g}_2(w)\,z^2 + \mathsf{g}_1(w)\,z + \mathsf{g}_0(w) = 0 \end{cases} \tag{24}$$

with

$$\begin{aligned} \mathsf{f}_0(w) &= p(1+E\cos\alpha)\left\langle \mathcal{P}\sin\alpha - \mathcal{Q}(E+\cos\alpha), (1-w^2)\mathfrak{a} + 2w\mathfrak{b}\right\rangle + EP\sin\alpha\, f_e^\beta(w) ; \\ \mathsf{f}_1(w) &= 2p\left\langle \mathcal{P}[\cos\alpha + E(\cos^2\alpha - \sin^2\alpha)] + \mathcal{Q}\sin\alpha(1+2E\cos\alpha+E^2), (1-w^2)\mathfrak{a} + 2w\mathfrak{b}\right\rangle + \\ &\quad +2EP\cos\alpha\, f_e^\beta(w) ; \\ \mathsf{f}_2(w) &= -6pE\sin\alpha\left\langle \mathcal{P}\cos\alpha + \mathcal{Q}\sin\alpha, (1-w^2)\mathfrak{a} + 2w\mathfrak{b}\right\rangle ; \\ \mathsf{f}_3(w) &= 2p\left\langle \mathcal{P}[\cos\alpha - E(\cos^2\alpha - \sin^2\alpha)] + \mathcal{Q}\sin\alpha(1-2E\cos\alpha+E^2), (1-w^2)\mathfrak{a} + 2w\mathfrak{b}\right\rangle + \end{aligned}$$

$$
\begin{aligned}
&\quad +2EP\cos\alpha\, f_e^\beta(w)\,;\\
\mathsf{f}_4(w) &= -p(1-E\cos\alpha)\left\langle \mathcal{P}\sin\alpha + \mathcal{Q}(E-\cos\alpha), (1-w^2)\mathfrak{a} + 2w\mathfrak{b}\right\rangle - EP\sin\alpha\, f_e^\beta(w)\,;
\end{aligned}
$$

and

$$
\begin{aligned}
\mathsf{g}_0(w) &= P\, f_e^\beta(w)\left\langle \mathfrak{t}_e^\beta(w), \mathcal{A}\right\rangle + ep(1+E\cos\alpha)(1+w^2)\left[-\sin\beta w^2 + 2\cos\beta w + \sin\beta\right]\,;\\
\mathsf{g}_1(w) &= 2P\, f_e^\beta(w)\left\langle \mathfrak{t}_e^\beta(w), \mathcal{B}\right\rangle - 2epE\sin\alpha(1+w^2)\left[-\sin\beta w^2 + 2\cos\beta w + \sin\beta\right]\,;\\
\mathsf{g}_2(w) &= -P\, f_e^\beta(w)\left\langle \mathfrak{t}_e^\beta(w), \mathcal{A}\right\rangle + ep(1-E\cos\alpha)(1+w^2)\left[-\sin\beta w^2 + 2\cos\beta w + \sin\beta\right]\,.
\end{aligned}
$$

where we have introduced the scalar factor

$$
f_e^\beta(w) = \Big[(1-e\cos\beta)\, w^2 - 2we\sin\beta + (1+e\cos\beta)\Big]\,,
$$

and the vector

$$
\mathfrak{t}_e^\beta(w) = \mathfrak{p}\left[-w^2\sin\beta + 2w\cos\beta + \sin\beta\right] - \mathfrak{q}\left[(e+\cos\beta)w^2 - 2w\sin\beta + (e-\cos\beta)\right]\,.
$$

Remark: By (13) we have sent to infinity the points with the V component equal to $\pi+\alpha$ and the points with the v component equal to $\pi+\beta$.

10.2. Elimination of the variable z and factorization of the resultant

The resultant $\mathsf{Res}_{\alpha,\beta}(w) = \mathtt{Res}(\mathsf{f}_{\alpha,\beta}(z,w), \mathsf{g}_{\alpha,\beta}(z,w), z)$ is given by the determinant of the Sylvester matrix

$$
\mathsf{S}_{\alpha,\beta}(w) = \begin{pmatrix}
\mathsf{f}_4 & 0 & \mathsf{g}_2 & 0 & 0 & 0\\
\mathsf{f}_3 & \mathsf{f}_4 & \mathsf{g}_1 & \mathsf{g}_2 & 0 & 0\\
\mathsf{f}_2 & \mathsf{f}_3 & \mathsf{g}_0 & \mathsf{g}_1 & \mathsf{g}_2 & 0\\
\mathsf{f}_1 & \mathsf{f}_2 & 0 & \mathsf{g}_0 & \mathsf{g}_1 & \mathsf{g}_2\\
\mathsf{f}_0 & \mathsf{f}_1 & 0 & 0 & \mathsf{g}_0 & \mathsf{g}_1\\
0 & \mathsf{f}_0 & 0 & 0 & 0 & \mathsf{g}_0
\end{pmatrix}\,;
$$

it is generically a 20-th degree polynomial in the variable w.

We want to use the basic properties of the determinants to extract the factor $f_e^\beta(w)$ from the resultant. Let us define the following terms:

$$
\mathcal{A}_E^\alpha = \frac{E\sin\alpha}{1+E\cos\alpha}\,; \qquad \mathcal{C}_E^\alpha = \frac{E^2-1+E^2\sin^2\alpha}{(1+E\cos\alpha)^2}\,;
$$

$$
\mathcal{B}_E^\alpha = \frac{E\sin\alpha}{1-E\cos\alpha}\,; \qquad \mathcal{D}_E^\alpha = \frac{E^2-1+E^2\sin^2\alpha}{(1-E\cos\alpha)^2}\,;
$$

$$\mathcal{E}_E^\alpha = \frac{E\sin\alpha}{(1+E\cos\alpha)^3}\Big[3(E^2-1)+E^2\sin^2\alpha\Big] \; ;$$
$$\mathcal{F}_E^\alpha = \frac{E\sin\alpha}{(1-E\cos\alpha)^3}\Big[3(E^2-1)+E^2\sin^2\alpha\Big] \; .$$

We perform these operations on the rows of $\mathsf{S}_{\alpha,\beta}$ to factorize the resultant $\mathsf{Res}_{\alpha,\beta}(w)$:

1. substitute the 3^{rd} row with the linear combination

$$(3^{rd}\text{ row}) + \mathcal{B}_E^\alpha\,(2^{nd}\text{ row}) + \mathcal{A}_E^\alpha\,(4^{th}\text{ row}) + \mathcal{D}_E^\alpha\,(1^{st}\text{ row}) + \\ + \; \mathcal{C}_E^\alpha\,(5^{th}\text{ row}) + \mathcal{E}_E^\alpha\,(6^{th}\text{ row})\,;$$

2. substitute the 4th row with the linear combination

$$(4^{th}\text{ row}) + \mathcal{B}_E^\alpha\,(3^{rd}\text{ row}) + \mathcal{A}_E^\alpha\,(5^{th}\text{ row}) + \mathcal{D}_E^\alpha\,(2^{nd}\text{ row}) + \\ + \; \mathcal{C}_E^\alpha\,(6^{th}\text{ row}) + \mathcal{F}_E^\alpha\,(1^{st}\text{ row})\,.$$

We obtain the matrix

$$\tilde{\mathsf{S}}_{\alpha,\beta}(w) = \begin{pmatrix} \mathsf{f}_4 & 0 & \mathsf{g}_2 & 0 & 0 & 0 \\ \mathsf{f}_3 & \mathsf{f}_4 & \mathsf{g}_1 & \mathsf{g}_2 & 0 & 0 \\ \mathsf{r}_{3,1} & \mathsf{r}_{3,2} & \mathsf{r}_{3,3} & \mathsf{r}_{3,4} & \mathsf{r}_{3,5} & \mathsf{r}_{3,6} \\ \mathsf{r}_{4,1} & \mathsf{r}_{4,2} & \mathsf{r}_{4,3} & \mathsf{r}_{4,4} & \mathsf{r}_{4,5} & \mathsf{r}_{4,6} \\ \mathsf{f}_0 & \mathsf{f}_1 & 0 & 0 & \mathsf{g}_0 & \mathsf{g}_1 \\ 0 & \mathsf{f}_0 & 0 & 0 & 0 & \mathsf{g}_0 \end{pmatrix} \; ;$$

where

$$\begin{aligned}
\mathsf{r}_{3,1}(w) &= \mathsf{f}_2(w) + \mathcal{B}_E^\alpha\mathsf{f}_3(w) + \mathcal{A}_E^\alpha\mathsf{f}_1(w) + \mathcal{D}_E^\alpha\mathsf{f}_4(w) + \mathcal{C}_E^\alpha\mathsf{f}_0(w) = f_e^\beta(w)\,\tilde{\mathsf{r}}_{3,1}\,; \\
\mathsf{r}_{3,2}(w) &= \mathsf{f}_3(w) + \mathcal{B}_E^\alpha\mathsf{f}_4(w) + \mathcal{A}_E^\alpha\mathsf{f}_2(w) + \mathcal{C}_E^\alpha\mathsf{f}_1(w) + \mathcal{E}_E^\alpha\mathsf{f}_0(w) = f_e^\beta(w)\,\tilde{\mathsf{r}}_{3,2}\,; \\
\mathsf{r}_{3,3}(w) &= \mathsf{g}_0(w) + \mathcal{B}_E^\alpha\mathsf{g}_1(w) + \mathcal{D}_E^\alpha\mathsf{g}_2(w) = f_e^\beta(w)\,\tilde{\mathsf{r}}_{3,3}(w)\,; \\
\mathsf{r}_{3,4}(w) &= \mathsf{g}_1(w) + \mathcal{B}_E^\alpha\mathsf{g}_2(w) + \mathcal{A}_E^\alpha\mathsf{g}_0(w) = f_e^\beta(w)\,\tilde{\mathsf{r}}_{3,4}(w)\,; \\
\mathsf{r}_{3,5}(w) &= \mathsf{g}_2(w) + \mathcal{A}_E^\alpha\mathsf{g}_1(w) + \mathcal{C}_E^\alpha\mathsf{g}_0(w) = f_e^\beta(w)\,\tilde{\mathsf{r}}_{3,5}(w)\,; \\
\mathsf{r}_{3,6}(w) &= \mathcal{A}_E^\alpha\mathsf{g}_2(w) + \mathcal{C}_E^\alpha\mathsf{g}_1(w) + \mathcal{E}_E^\alpha\mathsf{g}_0(w) = f_e^\beta(w)\,\tilde{\mathsf{r}}_{3,6}(w)\,; \\
\mathsf{r}_{4,1}(w) &= \mathsf{f}_1(w) + \mathcal{B}_E^\alpha\mathsf{f}_2(w) + \mathcal{A}_E^\alpha\mathsf{f}_0(w) + \mathcal{D}_E^\alpha\mathsf{f}_3(w) + \mathcal{F}_E^\alpha\mathsf{f}_4(w) = f_e^\beta(w)\,\tilde{\mathsf{r}}_{4,1}\,; \\
\mathsf{r}_{4,2}(w) &= \mathsf{r}_{3,1}(w)\,; \\
\mathsf{r}_{4,3}(w) &= \mathcal{B}_E^\alpha\mathsf{g}_0(w) + \mathcal{D}_E^\alpha\mathsf{g}_1(w) + \mathcal{F}_E^\alpha\mathsf{g}_2(w) = f_e^\beta(w)\,\tilde{\mathsf{r}}_{4,3}(w)\,; \\
\mathsf{r}_{4,4}(w) &= \mathsf{r}_{3,3}(w)\,; \\
\mathsf{r}_{4,5}(w) &= \mathsf{r}_{3,4}(w)\,; \\
\mathsf{r}_{4,6}(w) &= \mathsf{r}_{3,5}(w)\,;
\end{aligned}$$

for some polynomials $\tilde{\mathsf{r}}_{i,j}$. It follows that

$$\mathsf{Res}_{\alpha,\beta}(w) = det(\tilde{\mathsf{S}}_{\alpha,\beta}(w)) = \Big[f_e^\beta(w)\Big]^2\; det(\hat{\mathsf{S}}_{\alpha,\beta}(t))$$

with

$$\hat{\mathsf{S}}_{\alpha,\beta}(w) = \begin{pmatrix} \mathsf{f}_4 & 0 & \mathsf{g}_2 & 0 & 0 & 0 \\ \mathsf{f}_3 & \mathsf{f}_4 & \mathsf{g}_1 & \mathsf{g}_2 & 0 & 0 \\ \tilde{\mathsf{r}}_{3,1} & \tilde{\mathsf{r}}_{3,2} & \tilde{\mathsf{r}}_{3,3} & \tilde{\mathsf{r}}_{3,4} & \tilde{\mathsf{r}}_{3,5} & \tilde{\mathsf{r}}_{3,6} \\ \tilde{\mathsf{r}}_{4,1} & \tilde{\mathsf{r}}_{4,2} & \tilde{\mathsf{r}}_{4,3} & \tilde{\mathsf{r}}_{4,4} & \tilde{\mathsf{r}}_{4,5} & \tilde{\mathsf{r}}_{4,6} \\ \mathsf{f}_0 & \mathsf{f}_1 & 0 & 0 & \mathsf{g}_0 & \mathsf{g}_1 \\ 0 & \mathsf{f}_0 & 0 & 0 & 0 & \mathsf{g}_0 \end{pmatrix} .$$

REMARK: The factor $f_e^\beta(w)$ that can be collected with multiplicity 2 from the resultant $\mathsf{Res}_{\alpha,\beta}(w)$, corresponds to the term $1 + e\cos(\xi + \beta)$ in (12) and has the roots

$$w_{1,2} = \frac{e \sin\beta \pm \sqrt{e^2 - 1}}{1 - e\cos\beta} .$$

We can apply the strategy described in Subsections 4.3, 4.4 to compute the roots of system (24).

10.3. Some particular cases

Case $e = 0$

The second equation in (12) gives us

$$\begin{aligned} &\langle \mathfrak{p}\sin(\xi+\beta) - \mathfrak{q}\cos(\xi+\beta), \mathcal{A}\cos\Xi + \mathcal{B}\sin\Xi \rangle = \\ &= \langle \mathfrak{a}\sin\xi - \mathfrak{b}\cos\xi, \mathcal{A}\cos\Xi + \mathcal{B}\sin\Xi\rangle = 0 \end{aligned}$$

so that, applying the variable change (13), we obtain

$$(w^2-1)\left[\langle \mathfrak{b}, \mathcal{A}\rangle(1-z^2) + 2\langle \mathfrak{b}, \mathcal{B}\rangle z\right] + 2w\left[\langle \mathfrak{a}, \mathcal{A}\rangle(1-z^2) + 2\langle \mathfrak{a}, \mathcal{B}\rangle z\right] = 0$$

whose degree in the variable w has decreased from 4 to 2 with respect to the general case.

Case $E = 0$

By a symmetry argument, applying (13) to the first equation in (12) we obtain

$$(z^2-1)\left[\langle \mathcal{B}, \mathfrak{a}\rangle(1-w^2) + 2\langle \mathcal{B}, \mathfrak{b}\rangle w\right] + 2z\left[\langle \mathcal{A}, \mathfrak{a}\rangle(1-w^2) + 2\langle \mathcal{A}, \mathfrak{b}\rangle w\right] = 0$$

whose degree in the variable z has decreased from 4 to 2 with respect to the general case.

Case $e = 1$

The second equation in (12) can be written as

$$\begin{aligned} &P\left[1+\cos(\xi+\beta)\right]\langle \mathfrak{p}\sin(\xi+\beta) - \mathfrak{q}\left[1+\cos(\xi+\beta)\right], \mathcal{A}\sin\Xi + \mathcal{B}\cos\Xi\rangle + \\ &+ p\sin(\xi+\beta)\left[1 + E\cos(\Xi+\alpha)\right] = 0 . \end{aligned}$$

We observe that

$$\begin{cases} 1+\cos(\xi+\beta) = (1-\cos\beta)\dfrac{(w-w_+)^2}{1+w^2} \\ \sin(\xi+\beta) = -\sin\beta\dfrac{(w-w_+)(w-w_-)}{1+w^2} \end{cases} \quad \text{where} \quad \begin{cases} w_+ = \dfrac{\cos\beta+1}{\sin\beta} \\ w_- = \dfrac{\cos\beta-1}{\sin\beta} \end{cases}$$

so that each of the terms $\mathsf{g}_0(w), \mathsf{g}_1(w), \mathsf{g}_2(w)$ in (24) has in this case a factor $(w-w_+)$. Applying the linear combinations used to compute the matrix $\tilde{\mathsf{S}}_{\alpha,\beta}(w)$ we obtain a factor $f_1^\beta(w) = (1-\cos\beta)(w-w_+)^2$: thus, using the basic properties of determinants, we can extract a factor $(w-w_+)^8$ from the resultant $\mathsf{Res}_{\alpha,\beta}(w)$. These solutions have to be discarded because they correspond to points at infinity on the parabolic orbit, as we can check by passing to the limit for $\beta \to 0$. Note that in this case the application of the variable change (13) with $\beta = 0$ prevents from searching just this point.

CASE $E = 1$

By a symmetry argument we can prove that the value of z corresponding to

$$z_+ = \frac{\cos\alpha + 1}{\sin\alpha}$$

is a root with multiplicity 8 of $\mathsf{Res}^*_{\alpha,\beta}(z) = \mathtt{Res}(\mathsf{f}_{\alpha,\beta}, \mathsf{g}_{\alpha,\beta}, w)(z)$, that is the resultant of the polynomials $\mathsf{f}_{\alpha,\beta}, \mathsf{g}_{\alpha,\beta}$ with respect to the other variable w. These roots have also to be discarded.

Note that using the angular shifts we can avoid a degenerate case discussed before: we can select values α, β such that the degrees of $\mathsf{f}_{\alpha,\beta}, \mathsf{g}_{\alpha,\beta}$ as polynomials in the variable z are 4 and 2 respectively also for $E = 1$. This allows to compute $\mathsf{Res}_{\alpha,\beta}(w)$ as the determinant of the 6×6 matrix $\mathsf{S}_{\alpha,\beta}(w)$ also in this case.

10.4. Pairs of real solutions

We shall give a simple example of a polynomial system of two equations in two variables, with real coefficients, such that the resultant computed with respect to different variables gives a different number of real solutions. Let us consider the system

$$\begin{cases} u(v^2+1) = 0 \\ v(u^2-1) = 0 \end{cases} ;$$

the resultant with respect to v is

$$det \begin{bmatrix} u & (u^2-1) & 0 \\ 0 & 0 & (u^2-1) \\ u & 0 & 0 \end{bmatrix} = u(u^2-1)^2 \,,$$

while the resultant with respect to u is

$$det \begin{bmatrix} v & (v^2+1) & 0 \\ 0 & 0 & (v^2+1) \\ -v & 0 & 0 \end{bmatrix} = -v(v^2+1)^2 = 0 \ .$$

Thus we have 3 real solutions $u = 0, 1, -1$ (the last two with multiplicity 2 each) for the first equation, and only one real solution $v = 0$ for the second (the other solutions are $v = \pm i$ with multiplicity 2 each).

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