# On the uncertainty of the minimal distance between two confocal Keplerian orbits

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**Abstract**

We introduce a regularization for the minimal distance maps, giving the locally minimal values of the distance between two points on two confocal Keplerian orbits. This allows to define a meaningful uncertainty for the minimal distance also when orbit crossings are possible, and it is useful to detect the possibility of collisions or close approaches between two celestial bodies moving approximatively on these orbits, with important consequences in the study of their dynamics. An application to the orbit of a recently discovered near–Earth asteroid is also given.

**Keywords:** regularization, orbit distance, covariance propagation

## 1 Introduction

The *orbit distance* between two Keplerian orbits[1], solutions of the Kepler problem with a common center of force, is useful to know if two celestial bodies moving along these orbits can collide or undergo a very close approach: if the orbit distance is large enough there is no possibility of such an event, at least during the time span in which the Keplerian solutions are a good approximation of the real orbits.

When a new celestial body is discovered we do not know its orbit at first, in fact we can only observe the positions of that body from another one at different times, typically from the Earth or from an orbiting station, like the Hubble space telescope. The observations of the body allow to compute its orbit by means of the orbit determination methods, e.g. Gauss' or Laplace's method (see [14]). Nevertheless the observations are affected by errors due to different reasons[2] and already in classical orbit determination methods the orbit was given with an uncertainty, accounting for the presence of observational errors (see [8]).

The orbit uncertainty of recently discovered asteroids (or comets) is typically large, so that we usually have a large region of possible initial conditions

[1] the distance between the two orbits as geometrical sets

[2] instrumental errors, atmospheric turbulence imperfections, star catalog errors and so on

to be used to predict the motion. Furthermore the motion of such bodies can be regular or chaotic, even strongly, and the large uncertainty in the orbit determination makes the study of their dynamics a particularly difficult problem, with different possible features.

As long as an asteroid has no close approaches with a massive body of the Solar system, like a planet, its Keplerian heliocentric orbit (solution of a two-body problem with the Sun) is a good approximation of the path followed by the asteroid for a comparatively short time span. In this case we can compute the asteroid evolution for a longer time span by the classical perturbation theory (see [17]), that takes into account the effects on the Keplerian orbit due to the presence of the planets.

On the other hand, if there are close approaches with a planet, the asteroid is strongly perturbed and its dynamics can even be dominated by the gravitational attraction of the other approaching body, as in the case of an Earth-crossing asteroid. When an asteroid undergoes a very close encounter with our planet its orbit can be well approximated, during the encounter, by a branch of a *geocentric* Keplerian orbit; the initially computed heliocentric orbit is largely modified by the encounter, to the point that the actual path of the asteroid cannot be computed by means of the perturbation theory (see [18]). Thus the study of some dynamical properties of the asteroids, like their chaotic or regular behavior, is strictly dependent on the occurrence of close approaches, that can be excluded by an estimate of the orbit distance; actually the computation of the latter should precede the study of the asteroid dynamics.

Once an orbit of an asteroid and its uncertainty are given, additional observations make the uncertainty smaller, but the difficulty itself to follow the body up in the sky is related to the occurrence of close approaches with the planets.

Furthermore it is useful to produce and update an observation priority list, based on the possible orbit distance with the Earth, that shows the astronomers which are the asteroids that should be followed up with particular care.

For all these reasons a mathematical theory to compute the uncertainty of the orbit distance is an important tool for the scientific applications.

A simple geometric consideration suggests that two confocal orbits may get close at more than one pair of points, thus it is necessary to compute not only the absolute minimum, called MOID[3] in the literature, of the distance function $d$ between two points along the orbits, but all its local minimum values. We can easily obtain these values by computing all the critical points of the function $d^2$, squared to be smooth also at the orbit crossing points.

An approximation of the orbit distance can be easily computed by densely sampling the two orbits and comparing the values of the distance $d$ at the pairs of sampled points. We can use the corresponding approximation of the absolute minimum point as a starting guess of an iterative method, like Newton-Raphson's, to obtain a more precise result by computing the solution of the critical points equation (1), as in [15]. We can also follow a similar procedure for the other local minimum points, but this algorithm may converge to a wrong

[3] *Minimal Orbit Intersection Distance*

critical point if the starting guess is not close enough to the desired solution. The possibility of a wrong choice for the starting guess of an iterative method appears more evident by considering the maximum number of critical points that can occur in this problem: in [9], [10] there are examples with up to twelve critical points and up to four local minima of $d$. The critical points may even be infinitely many, but only in very rare cases, completely determined in [10].

Recently some new methods have been proposed to compute the critical points of $d^2$ using tools from the algebraic elimination theory (see [12], [9], [10]). These methods deal with the problem in a polynomial form and the solutions can be computed more easily and in a more reliable way than by solving general nonlinear equations. Nevertheless, even overcoming the difficulties arising in the computation of the critical points of $d^2$, additional problems appear when we take into account the uncertainty in the knowledge of the orbits.

The uncertainty of the orbit distance can be generically computed by the uncertainty propagation formula (24), but the possibility of orbit crossings produces a singularity in this computation. A first difficulty is that relation (24) involves the partial derivatives of the orbit distance with respect to the elements, but these derivatives do not exist when the two orbits intersect each other (see Section 3). A trivial regularization of the distance, allowing the existence of these derivatives, is given by the squared orbit distance $d^2_{min}$ (it is a regular map of the elements also where it vanishes), or by *Plummer's softening* $\sqrt{d^2_{min} + \epsilon^2}$ with a small real number $\epsilon$, as used in [2] to regularize the two-body potential. Nevertheless the partial derivatives of these regularized distances with respect to the orbital elements vanish at crossing configurations, thus we cannot use the uncertainty propagation formula (24) as well.

An additional (and more worrying) difficulty is that the uncertainty of a non-zero but small orbit distance may allow negative values of the distance, that are meaningless. Both these problems are particularly unpleasant because we would like to know the uncertainty just when the orbit distance can be small or vanishing, that is when a collision or a close approach is possible.

A solution has been proposed in [3] by computing an approximation for the MOID: in several cases the local minima of the distance $d$ occur close to the mutual nodes, intersections of the line common to both orbital planes with the orbits themselves (see Appendix 8.2). Thus we can consider the straight lines representing a linearization of the orbits at the mutual nodes (both ascending and descending) and take the distance between these lines as two approximations of the local minima. It is also possible to give a sign to these distances and then to overcome the difficulties explained before.

The problems with this approach are the following: 1) if the mutual orbital inclination $I_M$ is zero the mutual nodes are not defined, 2) for small values of $I_M$ the minimum points are usually not close to the mutual nodes 3) the approximations of the local minima at the mutual nodes cannot be more than two, while there are known cases with up to four local minimum points (see [9], [10]).

The purpose of this paper is to define regular maps, functions of the parameters defining the two–orbit configurations, giving the local minima of $d$

without any approximations and allowing negative values; using these maps we can compute a meaningful uncertainty of the orbit distance. The main idea is to perform a suitable *cut–off* of the two–orbit configuration domain that allow to change the sign of the distance on selected subsets of the remaining domain in a way that the resulting map is analytic.

In Section 2 we introduce the Keplerian distance function $d$ and the minimal distance maps related to the critical points of $d^2$; we also discuss the singularities of these maps. Section 3 is devoted to the regularization of the singularity given by the vanishing of the orbit distance. The computation of the uncertainty of the minimal distance, based on the uncertainty in the determination of the orbits, is explained in Section 4, where we also propose to use different definitions of the distance according to the singularities that the uncertainty of the orbits may lead to. We conclude this work by discussing the case of a recently discovered near–Earth asteroid, Apophis (99942), that has been carefully followed up by the astronomers to exclude the possibility of an impact with the Earth in the next future.

# 2 The Keplerian distance function and its critical points

Let us consider two Keplerian orbits with a common center of force. As it is well known, these orbits are conics, bounded (circles, ellipses) or unbounded (parabolas, hyperbolas)[4]. They have a common focus as they share the same center of force.

Different sets of orbital elements, e.g. Keplerian, equinoctial or cometary (see [4] and Appendix 8.2, 8.3) can be used to describe them. We shall adopt a general point of view, allowing different choices of the orbital elements: let us consider a set $\mathcal{E} = (E_1, E_2)$ of 10 elements, composed by two subsets of 5 elements each, such that $E_r$ defines the geometric configuration of the $r$-th orbit ($r = 1, 2$). Furthermore we consider the vector $V = (v_1, v_2)$, consisting of two parameters along the orbits. For example, one possible choice for $\mathcal{E}$ is $E_1 = (q_1, e_1, i_1, \Omega_1, \omega_1), E_2 = (q_2, e_2, i_2, \Omega_2, \omega_2)$ where $q_r$ are the pericenter distances, $e_r$ the eccentricities, $i_r$ the inclinations, $\Omega_r$ the longitudes of the ascending node and $\omega_r$ the pericenter arguments; then we can select $(v_1, v_2)$ as the vector of the true anomalies.

Given a reference frame centered in the common focus, let $\mathcal{X}_1 = \mathcal{X}_1(E_1, v_1)$, $\mathcal{X}_2 = \mathcal{X}_2(E_2, v_2) \in \mathbb{R}^3$ be the Cartesian coordinates of two bodies on the two orbits, with components $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$ respectively. Moreover we shall use $\langle \, , \, \rangle$ for the Euclidean scalar product.

We shall always assume that the orbital elements $(\mathcal{E}, V)$ satisfy the following basic regularity property:

$$\mathcal{X}_r \text{ is an analytic function of the elements } (E_r, v_r) \text{ for } r = 1, 2 \; .$$

[4] we shall not deal with the degenerate case of rectilinear orbits, possible only if the two-body angular momentum is zero and leading to collision with the central body

The elements traditionally used in Celestial mechanics and in Astronomy fulfill this property in a neighborhood of almost all the orbit configurations. Different elements are singular for different geometric configurations, hence it is convenient to have a general formulation of the problem allowing to change the choice of the orbital elements.

DEFINITION 1. *For each choice of the orbit parameters $\mathcal{E}$ we define the* Keplerian distance function $d$ *as the map*

$$\mathcal{V} \ni V \mapsto d(\mathcal{E}, V) \stackrel{def}{=} \sqrt{\langle \mathcal{X}_1 - \mathcal{X}_2, \mathcal{X}_1 - \mathcal{X}_2 \rangle} \in \mathbb{R}^+ \ ,$$

*where $\mathcal{V} = \mathbb{T}^2 = S^1 \times S^1$ (a two–dimensional torus) if both orbits are bounded, $\mathcal{V} = S^1 \times \mathbb{R}$ (an infinite cylinder) if only one is bounded, and $\mathcal{V} = \mathbb{R} \times \mathbb{R}$ if they are both unbounded.*

In the following we shall denote the Keplerian distance function with $d(\mathcal{E}, \cdot)$ if we want to stress the dependence on the selected configuration $\mathcal{E}$.

Let $V_j(\mathcal{E}) = (v_1^{(j)}(\mathcal{E}), v_2^{(j)}(\mathcal{E}))$ be the values of the $j$-th critical point of $d^2(\mathcal{E}, \cdot)$, solution of

$$\nabla_V d^2(\mathcal{E}, V) = 0 \,, \tag{1}$$

with

$$\nabla_V d^2 = \left( \frac{\partial d^2}{\partial v_1}, \frac{\partial d^2}{\partial v_2} \right)^t \ ,$$

and let

$$\mathcal{X}_1^{(j)}(\mathcal{E}) = \mathcal{X}_1(E_1, v_1^{(j)}(\mathcal{E})) \, ; \qquad \mathcal{X}_2^{(j)}(\mathcal{E}) = \mathcal{X}_2(E_2, v_2^{(j)}(\mathcal{E}))$$

be the corresponding Cartesian coordinates. The number of critical points of $d^2$ is generically finite; in [10] it has been proved that they can be infinitely many only in the case of two coplanar (concentric) circles or two overlapping conics. Except for these two very peculiar cases, we can define the Keplerian distance at the $j$-th critical point of $d^2$ as

$$\begin{aligned} d_j(\mathcal{E}) &\stackrel{def}{=} d(\mathcal{E}, V_j(\mathcal{E})) = \\ &= \sqrt{\langle \mathcal{X}_1^{(j)}(\mathcal{E}) - \mathcal{X}_2^{(j)}(\mathcal{E}), \mathcal{X}_1^{(j)}(\mathcal{E}) - \mathcal{X}_2^{(j)}(\mathcal{E}) \rangle} \ . \end{aligned}$$

DEFINITION 2. *Calling $\mathfrak{E}$ the* two–orbit configuration space*, locally homeomorphic to $\mathbb{R}^{10}$, we define the maps*

$$\mathfrak{E} \ni \mathcal{E} \mapsto V_j(\mathcal{E}) \in \mathcal{V} \, ; \qquad \mathfrak{E} \ni \mathcal{E} \mapsto d_j(\mathcal{E}) \in \mathbb{R}^+ \, ,$$

*representing the $j$-th critical point of $d^2(\mathcal{E}, \cdot)$ and the corresponding value of the distance for a given configuration $\mathcal{E}$.*

Of course the number itself of critical points, and then the range of the index $j$, may vary with the selected configuration; however this number can change at a configuration $\mathcal{E}_*$ only if the Hessian matrix of the squared Keplerian distance $d^2(\mathcal{E}_*, \cdot)$ is degenerate at some critical point of $d^2(\mathcal{E}_*, \cdot)$ (see [5]), that is if

$$\det \mathcal{H}_V(d^2)(\mathcal{E}_*, V_j(\mathcal{E}_*)) = 0\,, \tag{2}$$

where

$$\mathcal{H}_V(d^2) \quad = \quad \begin{pmatrix} \dfrac{\partial^2 d^2}{\partial v_1^2} & \dfrac{\partial^2 d^2}{\partial v_2 \partial v_1} \\ \\ \dfrac{\partial^2 d^2}{\partial v_1 \partial v_2} & \dfrac{\partial^2 d^2}{\partial v_2^2} \end{pmatrix}$$

is the Hessian matrix of $d^2(\mathcal{E}, \cdot)$. Thus, if the non–degeneracy condition

$$\det \mathcal{H}_V(d^2)(\bar{\mathcal{E}}, V_j(\bar{\mathcal{E}})) \neq 0 \tag{3}$$

holds for a given configuration $\bar{\mathcal{E}}$ and for every index $j$ of the critical points of $d^2(\bar{\mathcal{E}}, \cdot)$, then there exists an open neighborhood $\mathfrak{U} \subset \mathfrak{E}$ of $\bar{\mathcal{E}}$ such that the number of critical points of $d^2(\mathcal{E}, \cdot)$ is the same for each $\mathcal{E} \in \mathfrak{U}$. We can define the maps $V_j$ and $d_j$ in the neighborhood $\mathfrak{U}$ for every index $j$ of such critical points. Moreover we can choose $\mathfrak{U}$ and the order of the critical points in a way that each map $V_j$ is analytic[5]: this follows from the implicit function theorem applied to the critical points equation

$$\nabla_V d^2(\mathcal{E}, V) = 0,$$

where $\nabla_V d^2$ is a real analytic function (see [6]).
The partial derivatives of $V_j$ with respect to the element $\mathcal{E}_k$ at $\mathcal{E} \in \mathfrak{U}$ are given by

$$\frac{\partial V_j}{\partial \mathcal{E}_k}(\mathcal{E}) = -\left[\mathcal{H}_V(d^2)(\mathcal{E}, V_j(\mathcal{E}))\right]^{-1} \frac{\partial}{\partial \mathcal{E}_k} \nabla_V d^2(\mathcal{E}, V_j(\mathcal{E}))\,, \tag{4}$$

for $k = 1 \ldots 10$, where

$$\frac{\partial}{\partial \mathcal{E}_k} \nabla_V d^2 = \left( \frac{\partial^2 d^2}{\partial \mathcal{E}_k \partial v_1}, \frac{\partial^2 d^2}{\partial \mathcal{E}_k \partial v_2} \right)^t .$$

We shall be particularly interested in the local minimum points, corresponding to the subset of indexes $j_h$:

$$\mathcal{E} \mapsto d_{j_h}(\mathcal{E}) \stackrel{def}{=} d_h(\mathcal{E}) \qquad (\textit{locally minimal distance})\,. \tag{5}$$

From now on we shall use for brevity the index $h$ instead of $j_h$ to mean the quantities related to the local minima.

[5] indeed each map $V_j$ is defined on a different sheet of a Riemann surface that covers the two–orbit configuration space (see [16])

When at least one orbit is bounded we define the absolute minimum map[6]

$$\mathcal{E} \mapsto d_{min}(\mathcal{E}) \stackrel{def}{=} \min_h d_h(\mathcal{E}), \tag{6}$$

that for each two–orbit configuration returns the orbit distance.

The maps $d_h$ and $d_{min}$ just introduced have the following singularities:

i) $d_h$ and $d_{min}$ are not differentiable where they vanish (see Section 3);

ii) the absolute minimum point can be defined not univocally: when a two–orbit configuration $\bar{\mathcal{E}}$ admits two distinct points with the same value of $d(\bar{\mathcal{E}}, \cdot)$, that corresponds to the absolute minimum, then in a neighborhood of $\bar{\mathcal{E}}$ we can have two local minima that exchange their role as absolute minimum and $d_{min}$ can lose its regularity even without vanishing;

iii) when a bifurcation occurs the definition of the maps $d_h$ may become ambiguous after the bifurcation point. Note that this ambiguity does not occur for the $d_{min}$ map, that is defined by choosing the branch with the lowest value of the distance. The degeneration of the Hessian matrix of $d^2(\mathcal{E}, \cdot)$, leading to bifurcation phenomena, is also related to a loss of regularity of these maps (see relation (4)).

In Figure 1 we sketch the singularities just explained.

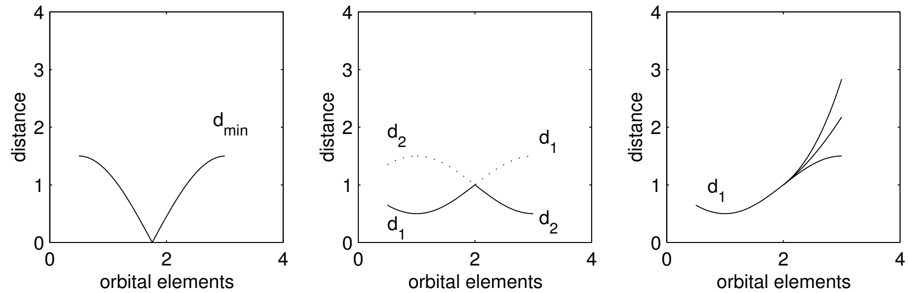


Figure 1: We sketch the three types of singularity of the maps $d_{min}$ and $d_h$. The figure on the left shows the loss of regularity corresponding to the vanishing of $d_{min}$ (a similar sketch is also valid for $d_h$). The figure in the middle shows the loss of regularity of the map $d_{min}$ ($= \min\{d_1, d_2\}$ in this case, drawn with solid line) due to the change of role between two local minima as absolute minimum (with a non–vanishing orbit distance). On the right we show a bifurcation of a local minimum into three points, e.g. two minimum points and a saddle: the map $d_1$ is not univocally defined after the bifurcation point.

In the following sections we shall study only the properties of the maps $d_h$, because the map $d_{min}$ has the additional singularity explained above in ii). In any case the knowledge of the minimal distances $d_h$ allows to determine also the orbit distance $d_{min}$.

[6]This hypothesis ensures the existence of the absolute minimum of the distance $d$. In the case of two unbounded orbits the infimum of $d$ may be reached at infinity, see [1].

# 3 Regularization of the minimal distance maps

In this section we shall prove that the maps $d_h$, defined in (5), are not regular functions of the orbital elements $\mathcal{E} = (\mathcal{E}_1, \ldots, \mathcal{E}_{10})$ where they vanish, but it is possible to remove this singularity by performing a suitable cut–off of its definition domain and changing the sign of these maps on selected subsets of the smaller resulting domain.[7]

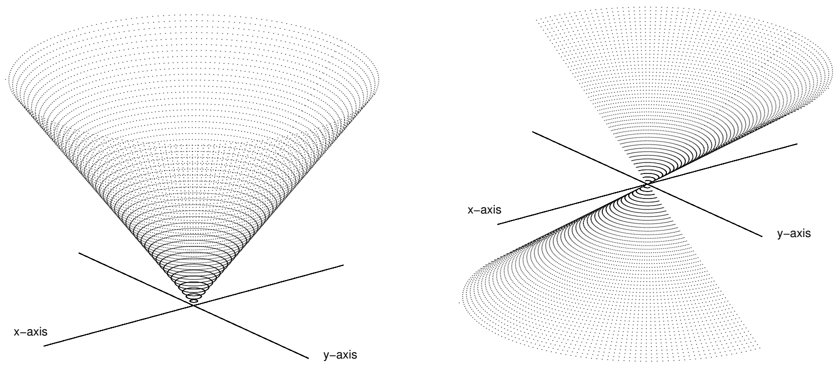


Figure 2: On the left we show the graphic of the function $f$: the directional derivatives at $(x, y) = (0, 0)$ do not exists for whatever choice of the direction. On the right we show the graphic of the regularized function $\tilde{f}$, extended to the origin $(0, 0)$ by continuity: in this case every directional derivative at $(x, y) = (0, 0)$ does exist.

We shall illustrate the basic idea of the regularization by a simple example: let us consider the function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$, defined as

$$f(x, y) = \sqrt{x^2 + y^2}\,;$$

its directional derivatives at $(x, y) = (0, 0)$ do not exist for every choice of the direction. We cut off the line $\{(x, y) \,|\, x = 0\}$ from the definition domain and change the sign of the function on the set $\{x > 0\}$: the result is the continuous function

$$\tilde{f}(x, y) = \begin{cases} -f(x, y) & \text{for } x > 0 \\ \phantom{-}f(x, y) & \text{for } x < 0 \end{cases} .$$

We can extend $\tilde{f}$ by continuity to the origin by setting $\tilde{f}(0, 0) = 0$, thus we obtain a function having all the directional derivatives at $(x, y) = (0, 0)$.

## 3.1 Derivatives of the minimal distance maps

Let us consider a minimal distance map $d_h : \mathfrak{U} \to \mathbb{R}^+$ and let $\bar{\mathcal{E}} \in \mathfrak{U}$ be a two–orbit configuration with $d_h(\bar{\mathcal{E}}) \neq 0$. The derivative of $d_h$ at $\bar{\mathcal{E}}$ with respect to

[7] the same results are also valid for the map $d_{min}$, apart maybe the configurations with two intersection points.

the orbital element $\mathcal{E}_k$ is given by

$$\frac{\partial d_h}{\partial \mathcal{E}_k}(\bar{\mathcal{E}}) = \frac{1}{2d_h(\bar{\mathcal{E}})}\frac{\partial d_h^2}{\partial \mathcal{E}_k}(\bar{\mathcal{E}}) \qquad \text{for } k = 1\dots 10\,,$$

where, using the chain rule,

$$\frac{\partial d_h^2}{\partial \mathcal{E}_k}(\bar{\mathcal{E}}) = \frac{\partial d^2}{\partial \mathcal{E}_k}(\bar{\mathcal{E}}, V_h(\bar{\mathcal{E}})) + \frac{\partial d^2}{\partial V}(\bar{\mathcal{E}}, V_h(\bar{\mathcal{E}}))\frac{\partial V_h}{\partial \mathcal{E}_k}(\bar{\mathcal{E}})$$

with

$$\frac{\partial V_h}{\partial \mathcal{E}_k}(\bar{\mathcal{E}}) = -\left[\mathcal{H}_V(d^2)(\bar{\mathcal{E}}, V_h(\bar{\mathcal{E}}))\right]^{-1}\frac{\partial}{\partial \mathcal{E}_k}\nabla_V d^2(\bar{\mathcal{E}}, V_h(\bar{\mathcal{E}}))\ .$$

Moreover we have

$$\frac{\partial d_h^2}{\partial \mathcal{E}_k}(\bar{\mathcal{E}}) = \frac{\partial d^2}{\partial \mathcal{E}_k}(\bar{\mathcal{E}}, V_h(\bar{\mathcal{E}}))\,, \tag{7}$$

in fact

$$\frac{\partial d^2}{\partial V}(\bar{\mathcal{E}}, V_h(\bar{\mathcal{E}})) = 0$$

because $V_h(\bar{\mathcal{E}})$ is a critical point of $d^2(\bar{\mathcal{E}}, \cdot)$.

Using (7) and the differences

$$\Delta = \mathcal{X}_1 - \mathcal{X}_2\,; \qquad \Delta_h = \mathcal{X}_1^{(h)} - \mathcal{X}_2^{(h)} \tag{8}$$

we can write

$$\frac{\partial d_h^2}{\partial \mathcal{E}_k}(\bar{\mathcal{E}}) = 2\left\langle \Delta_h(\bar{\mathcal{E}}, V_h(\bar{\mathcal{E}})), \frac{\partial \Delta}{\partial \mathcal{E}_k}(\bar{\mathcal{E}}, V_h(\bar{\mathcal{E}}))\right\rangle\ ,$$

so that, if $d_h(\bar{\mathcal{E}}) \neq 0$, we have

$$\frac{\partial d_h}{\partial \mathcal{E}_k}(\bar{\mathcal{E}}) = \left\langle \hat{\Delta}_h(\bar{\mathcal{E}}, V_h(\bar{\mathcal{E}})), \frac{\partial \Delta}{\partial \mathcal{E}_k}(\bar{\mathcal{E}}, V_h(\bar{\mathcal{E}}))\right\rangle \tag{9}$$

where

$$\hat{\Delta}_h = \frac{\Delta_h}{d_h} \tag{10}$$

is the unit vector map that, for each configuration $\mathcal{E}$, has the direction of the line joining the points on the two orbits that correspond to the local minimum point $V_h(\mathcal{E})$.

On the other hand, if $\bar{\mathcal{E}}$ is such that $d_h(\bar{\mathcal{E}}) = 0$, then expression (10) becomes singular and the limit of $\hat{\Delta}_h(\mathcal{E})$ for $\mathcal{E} \to \bar{\mathcal{E}}$ does not exist.

In the next paragraph we shall show that generically the direction (but *not* the orientation) of the unit vector $\hat{\Delta}_h$ is unique also in the limit $\mathcal{E} \to \bar{\mathcal{E}}$ with $d_h(\bar{\mathcal{E}}) = 0$. Intuitively this is due to a geometric characterization of the critical points of the squared distance function: the line joining two points on the curves that correspond to a critical point must be orthogonal to both tangent vectors to the curves at those points (see [10]). Thus, if the tangent vectors to the two orbits at the intersection point are independent they define univocally an orthogonal direction.

Nevertheless if $d_h(\bar{\mathcal{E}}) = 0$ the partial derivatives $\frac{\partial d_h}{\partial \mathcal{E}_k}(\bar{\mathcal{E}})$ generically do not exist; below we give a proof of this fact.

Let $\hat{e}_k, k = 1\dots 10$, be the unit vectors of the canonical basis in $\mathbb{R}^{10}$. First we need this result:

LEMMA 1. *Let $\bar{\mathcal{E}} \in \mathfrak{U}$ such that $d_h(\bar{\mathcal{E}}) = 0$. From the basic regularity property for each $k = 1\dots 10$ only the following two cases are possible:*

(i) $\exists\, \bar{\eta} > 0$ *such that* $d_h(\bar{\mathcal{E}} + \eta\hat{e}_k) \equiv 0, \forall\, \eta$ *with* $|\eta| \le \bar{\eta}$ *;*

(ii) $\exists\, \bar{\eta} > 0$ *such that* $d_h(\bar{\mathcal{E}} + \eta\hat{e}_k) \neq 0, \forall\, \eta$ *with* $0 < |\eta| \le \bar{\eta}$ *.*

*Proof.* It is a consequence of the analyticity of the maps $(\mathcal{E}, V) \mapsto \Delta(\mathcal{E}, V)$ and $\mathfrak{U} \ni \mathcal{E} \mapsto V_h(\mathcal{E})$, which implies that also the composite map $\mathbb{R} \ni \eta \mapsto \Delta_h(\bar{\mathcal{E}} + \eta\hat{e}_k) \in \mathbb{R}^3$ is analytic for $k = 1\dots 10$.

□

If for a given $k$ the case (i) of Lemma 1 holds, then the $k$-th partial derivative exists and is zero:

$$\frac{\partial d_h}{\partial \mathcal{E}_k}(\bar{\mathcal{E}}) = 0 \ .$$

On the other hand, if (ii) holds, we have $d_h(\bar{\mathcal{E}} + \eta\hat{e}_k) > 0$ for $|\eta| > 0$ small enough. Then by Lagrange's theorem the right and left partial derivatives of $d_h$ at $\bar{\mathcal{E}}$ are opposite, in fact we have

$$\begin{aligned} \frac{\partial^+ d_h}{\partial \mathcal{E}_k}(\bar{\mathcal{E}}) &= \lim_{\eta_1 \to 0^+} \frac{d_h(\bar{\mathcal{E}} + \eta_1\hat{e}_k) - d_h(\bar{\mathcal{E}})}{\eta_1} = \lim_{\eta_1 \to 0^+} \frac{\partial d_h}{\partial \mathcal{E}_k}(\xi_k(\eta_1)) = \\ &= -\lim_{\eta_2 \to 0^-} \frac{\partial d_h}{\partial \mathcal{E}_k}(\xi_k(\eta_2)) = -\lim_{\eta_2 \to 0^-} \frac{d_h(\bar{\mathcal{E}} + \eta_2\hat{e}_k) - d_h(\bar{\mathcal{E}})}{\eta_2} = -\frac{\partial^- d_h}{\partial \mathcal{E}_k}(\bar{\mathcal{E}}) \end{aligned}$$

where $\xi_k(\eta_1)$, $\xi_k(\eta_2)$ are two points on the segments joining $\bar{\mathcal{E}}$ and $\bar{\mathcal{E}} + \eta_r\hat{e}_k$, with $r = 1, 2$, and the partial derivative does not exist. The relation between the limits of the left and right partial derivatives is due to the representation formula (9) and to the following relation

$$\lim_{\eta \to 0^+} \hat{\Delta}_h(\bar{\mathcal{E}} + \eta\hat{e}_k) = -\lim_{\eta \to 0^-} \hat{\Delta}_h(\bar{\mathcal{E}} + \eta\hat{e}_k) \ .$$

As case (i) of Lemma 1 is a peculiar case, we obtain the generic non–existence of the partial derivatives $\frac{\partial d_h}{\partial \mathcal{E}_k}(\bar{\mathcal{E}})$.

### 3.2 Removal of the singularity

We can remove the singularity appearing in (9) for the configurations $\bar{\mathcal{E}} \in \mathfrak{U}$ such that $d_h(\bar{\mathcal{E}}) = 0$ by performing the following operations: first we choose a subset of the domain $\mathfrak{U}$ to cut–off, that properly contains the set

$$\{d_h = 0\} \stackrel{def}{=} \{\mathcal{E} \in \mathfrak{U} : d_h(\mathcal{E}) = 0\}\,;$$

then we change the sign of $\hat{\Delta}_h$ in different subsets of the smaller resulting domain $\mathfrak{U}_h$, depending on the selected minimum point index $h$. Finally, for each $\mathcal{E} \in \mathfrak{U}_h$, we give $d_h(\mathcal{E})$ the same sign as the one selected for $\hat{\Delta}_h(\mathcal{E})$ in the previous step and we show that the resulting function, called $\tilde{d}_h$, is continuous and continuously extendable to a wider domain $\tilde{\mathfrak{U}}_h$, that includes all the orbit crossings but the tangent ones (see Definition 3). In Proposition 2 we shall see that tangent crossing configurations correspond to the only crossing configurations where the Hessian matrix is degenerate at the crossing point. In Section 3.4 we shall prove that the maps $\tilde{d}_h$ are even analytic in $\tilde{\mathfrak{U}}_h$.

We start by looking for the portions of the domain $\mathfrak{U}$ to cut–off. For each two–orbit configuration $\mathcal{E} = (E_1, E_2) \in \mathfrak{U}$, the minimum point $V_h(\mathcal{E}) = (v_1^{(h)}(\mathcal{E}), v_2^{(h)}(\mathcal{E}))$ is in particular a critical point of $d^2(\mathcal{E}, \cdot)$, hence it must fulfill the relations

$$\begin{aligned} \frac{\partial d^2}{\partial v_1}(\mathcal{E}, V_h(\mathcal{E})) &= 2\ \langle\, \tau_1(\mathcal{E}), \Delta_h(\mathcal{E})\,\rangle = 0\,; \\ \frac{\partial d^2}{\partial v_2}(\mathcal{E}, V_h(\mathcal{E})) &= -2\ \langle\, \tau_2(\mathcal{E}), \Delta_h(\mathcal{E})\,\rangle = 0\,; \end{aligned} \tag{11}$$

where

$$\Delta_h = (\Delta_x^{(h)}, \Delta_y^{(h)}, \Delta_z^{(h)})\,;$$

$$\tau_1 = (\tau_{1,x}, \tau_{1,y}, \tau_{1,z})\,; \qquad \tau_2 = (\tau_{2,x}, \tau_{2,y}, \tau_{2,z})\,;$$

with

$$\begin{aligned} \tau_{1,x}(\mathcal{E}) &= \frac{\partial x_1}{\partial v_1}(E_1, v_1^{(h)}(\mathcal{E}))\,; \quad \tau_{2,x}(\mathcal{E}) = \frac{\partial x_2}{\partial v_2}(E_2, v_2^{(h)}(\mathcal{E}))\,; \\ \tau_{1,y}(\mathcal{E}) &= \frac{\partial y_1}{\partial v_1}(E_1, v_1^{(h)}(\mathcal{E}))\,; \quad \tau_{2,y}(\mathcal{E}) = \frac{\partial y_2}{\partial v_2}(E_2, v_2^{(h)}(\mathcal{E}))\,; \\ \tau_{1,z}(\mathcal{E}) &= \frac{\partial z_1}{\partial v_1}(E_1, v_1^{(h)}(\mathcal{E}))\,; \quad \tau_{2,z}(\mathcal{E}) = \frac{\partial z_2}{\partial v_2}(E_2, v_2^{(h)}(\mathcal{E}))\,. \end{aligned} \tag{12}$$

The vectors $\tau_1(\mathcal{E})$, $\tau_2(\mathcal{E})$ are tangent to the two orbits at the points $\mathcal{X}_1^{(h)}(\mathcal{E})$, $\mathcal{X}_2^{(h)}(\mathcal{E})$, corresponding to $V_h(\mathcal{E})$.

We define the matrixes maps

$$\mathcal{T} = \begin{pmatrix} \tau_{1,x} & \tau_{1,y} & \tau_{1,z} \\ \tau_{2,x} & \tau_{2,y} & \tau_{2,z} \end{pmatrix} \tag{13}$$

and

$$\mathcal{T}_1 = \begin{pmatrix} \tau_{1,y} & \tau_{1,z} \\ \tau_{2,y} & \tau_{2,z} \end{pmatrix}; \qquad \mathcal{T}_2 = \begin{pmatrix} \tau_{1,z} & \tau_{1,x} \\ \tau_{2,z} & \tau_{2,x} \end{pmatrix}; \qquad \mathcal{T}_3 = \begin{pmatrix} \tau_{1,x} & \tau_{1,y} \\ \tau_{2,x} & \tau_{2,y} \end{pmatrix}.$$

REMARK: The maps $\tau_1, \tau_2, \mathcal{T}, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ are defined on the domain $\mathfrak{U}$. For simplicity we have dropped the dependence on the index $h$.

DEFINITION 3. *We call* tangent crossing *configuration a two–orbit configuration $\bar{\mathcal{E}} \in \mathfrak{U}$ such that the two orbits intersect each other and the two tangent vectors $\tau_1(\bar{\mathcal{E}}), \tau_2(\bar{\mathcal{E}})$ at the intersection point are parallel (in symbols $\tau_1(\bar{\mathcal{E}}) \parallel \tau_2(\bar{\mathcal{E}})$). The intersection point is a minimum point $V_h(\bar{\mathcal{E}})$ of $d^2(\bar{\mathcal{E}}, \cdot)$, with minimum value $d_h(\bar{\mathcal{E}}) = 0$, we shall say that it is a tangent crossing point.*

Note that the existence of a tangent crossing point requires a zero mutual orbital inclination. In the Appendix 8.1 we show that two Keplerian orbits with a common focus cannot have more than two intersection points and when there is a tangent crossing point the intersection is only one.

REMARK: The matrix $\mathcal{T}(\mathcal{E})$ has rank $< 2$ if and only if the two tangent vectors $\tau_1(\mathcal{E}), \tau_2(\mathcal{E})$ are parallel. In case of orbit crossing the matrix $\mathcal{T}(\mathcal{E})$ has rank $< 2$ if and only if $\mathcal{E}$ is a tangent crossing configuration.

We introduce the maps

$$S_1 = \Delta_x^{(h)} \det(\mathcal{T}_1); \quad S_2 = \Delta_y^{(h)} \det(\mathcal{T}_2); \quad S_3 = \Delta_z^{(h)} \det(\mathcal{T}_3); \tag{14}$$

$$\tau_3 = \tau_1 \wedge \tau_2 = (\det(\mathcal{T}_1), \det(\mathcal{T}_2), \det(\mathcal{T}_3)) \tag{15}$$

and

$$\mathcal{W}(\mathcal{T}) = \sqrt{\det(\mathcal{T}_1)^2 + \det(\mathcal{T}_2)^2 + \det(\mathcal{T}_3)^2}; \tag{16}$$

they are all defined on the same domain $\mathfrak{U}$.

Let us consider a configuration $\mathcal{E} \in \mathfrak{U}$. If we assume that $S_1(\mathcal{E}) \neq 0$ we have $\Delta_x^{(h)}(\mathcal{E}) \neq 0$ and $\det(\mathcal{T}_1(\mathcal{E})) \neq 0$; then we can find a unique solution of the linear system

$$\begin{cases} \tau_{1,y}(\mathcal{E})\Delta_y^{(h)}(\mathcal{E}) + \tau_{1,z}(\mathcal{E})\Delta_z^{(h)}(\mathcal{E}) = -\tau_{1,x}(\mathcal{E})\Delta_x^{(h)}(\mathcal{E}) \\ \tau_{2,y}(\mathcal{E})\Delta_y^{(h)}(\mathcal{E}) + \tau_{2,z}(\mathcal{E})\Delta_z^{(h)}(\mathcal{E}) = -\tau_{2,x}(\mathcal{E})\Delta_x^{(h)}(\mathcal{E}) \end{cases} \tag{17}$$

with unknowns $\Delta_y^{(h)}(\mathcal{E})$ and $\Delta_z^{(h)}(\mathcal{E})$. The solutions are given by

$$\Delta_y^{(h)}(\mathcal{E}) = \Delta_x^{(h)}(\mathcal{E})\frac{\det(\mathcal{T}_2(\mathcal{E}))}{\det(\mathcal{T}_1(\mathcal{E}))}; \qquad \Delta_z^{(h)}(\mathcal{E}) = \Delta_x^{(h)}(\mathcal{E})\frac{\det(\mathcal{T}_3(\mathcal{E}))}{\det(\mathcal{T}_1(\mathcal{E}))}.$$

If $S_1(\mathcal{E}) \neq 0$, by substituting the solutions of the linear equations (17) in (10), we can write

$$\hat{\Delta}_h(\mathcal{E}) = \frac{|\det(\mathcal{T}_1(\mathcal{E}))|}{\det(\mathcal{T}_1(\mathcal{E}))} \frac{\Delta_x^{(h)}(\mathcal{E})}{|\Delta_x^{(h)}(\mathcal{E})|} \hat{\tau}_3(\mathcal{E}) \tag{18}$$

where

$$\hat{\tau}_3 = \frac{1}{\mathcal{W}(\mathcal{T})}(\det(\mathcal{T}_1),\ \det(\mathcal{T}_2),\ \det(\mathcal{T}_3))\ .$$

In a similar way if $S_2(\mathcal{E}) \neq 0$ we can write

$$\hat{\Delta}_h(\mathcal{E}) = \frac{|\det(\mathcal{T}_2(\mathcal{E}))|}{\det(\mathcal{T}_2(\mathcal{E}))}\frac{\Delta_y^{(h)}(\mathcal{E})}{|\Delta_y^{(h)}(\mathcal{E})|}\,\hat{\tau}_3(\mathcal{E}) \tag{19}$$

and, if $S_3(\mathcal{E}) \neq 0$,

$$\hat{\Delta}_h(\mathcal{E}) = \frac{|\det(\mathcal{T}_3(\mathcal{E}))|}{\det(\mathcal{T}_3(\mathcal{E}))}\frac{\Delta_z^{(h)}(\mathcal{E})}{|\Delta_z^{(h)}(\mathcal{E})|}\,\hat{\tau}_3(\mathcal{E})\ . \tag{20}$$

In conclusion, for each $\mathcal{E} \in \mathfrak{U}$, if at least one of the maps $S_1, S_2, S_3$ is different from zero in $\mathcal{E}$, we can write $\hat{\Delta}_h(\mathcal{E})$ in one of the form (18), (19), (20). We cut–off the set

$$\{S_1 = S_2 = S_3 = 0\} \stackrel{def}{=} \{\mathcal{E} \in \mathfrak{U} : S_1(\mathcal{E}) = S_2(\mathcal{E}) = S_3(\mathcal{E}) = 0\}$$

from the two–orbit configuration domain $\mathfrak{U}$ and define

$$\mathfrak{U}_h = \mathfrak{U} \setminus \{S_1 = S_2 = S_3 = 0\}\ .$$

We shall need the following results:

LEMMA 2. *We have these properties for the signs of the maps $S_1, S_2, S_3$:*

1. *If $S_1S_2 \neq 0$ then $\mathrm{sign}(S_1) = \mathrm{sign}(S_2)$;*
2. *If $S_1S_3 \neq 0$ then $\mathrm{sign}(S_1) = \mathrm{sign}(S_3)$;*
3. *If $S_2S_3 \neq 0$ then $\mathrm{sign}(S_2) = \mathrm{sign}(S_3)$ .*

*Proof.* These relations immediately follow from (18), (19) and (20).

□

The set

$$\{\mathcal{W}(\mathcal{T}) = 0\} \stackrel{def}{=} \{\mathcal{E} \in \mathfrak{U} : \mathcal{W}(\mathcal{T})(\mathcal{E}) = 0\}$$

corresponds to the configurations $\mathcal{E} \in \mathfrak{U}$ such that $\tau_1(\mathcal{E}), \tau_2(\mathcal{E})$ are parallel.

PROPOSITION 1. *The following relations hold*

*(i)* $\{\mathcal{W}(\mathcal{T}) = 0\} \subsetneq \{S_1 = S_2 = S_3 = 0\}$;

*(ii)* $\{d_h = 0\} \subsetneq \{S_1 = S_2 = S_3 = 0\}$;

*(iii)* $\{S_1 = S_2 = S_3 = 0\} \setminus \{\mathcal{W}(\mathcal{T}) = 0\} \subsetneq \{d_h = 0\}$.

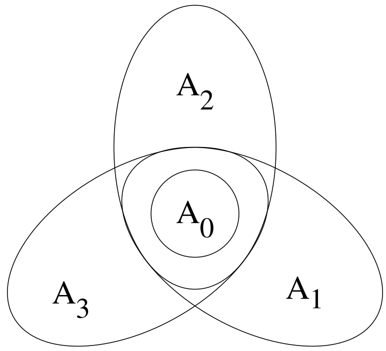


Figure 3: With the aid of an Euler-Venn diagram we describe the subsets of the two–orbit configuration space involved in the definition of $\tilde{d}_h$, given in (21). In this figure $A_1 = \{S_1 = 0\}$, $A_2 = \{S_2 = 0\}$, $A_3 = \{S_3 = 0\}$ and $A_0 = \{\mathcal{W}(\mathcal{T}) = 0\} \subsetneq A_1 \cap A_2 \cap A_3$.

*Proof.* The inclusions in $(i)$ and $(ii)$ are trivial; furthermore they are strict inclusions as the tangent vectors $\tau_1, \tau_2$ of a configuration $\mathcal{E}$ may be parallel without having $d_h(\mathcal{E}) = 0$ and vice versa. Let us prove the inclusion in $(iii)$: if $\mathcal{W}(\mathcal{T})(\mathcal{E}) \neq 0$ for a given $\mathcal{E} \in \mathfrak{U}$, then one of the components of $\tau_3(\mathcal{E})$ is nonzero. Assume $\det(\mathcal{T}_1(\mathcal{E})) \neq 0$: as $S_1(\mathcal{E}) = 0$ we must have $\Delta_x^{(h)}(\mathcal{E}) = 0$ and the linear system

$$\mathcal{T}_1(\mathcal{E}) \begin{pmatrix} \Delta_y^{(h)}(\mathcal{E}) \\ \Delta_z^{(h)}(\mathcal{E}) \end{pmatrix} = \begin{pmatrix} -\tau_{1,x}(\mathcal{E})\, \Delta_x^{(h)}(\mathcal{E}) \\ -\tau_{2,x}(\mathcal{E})\, \Delta_x^{(h)}(\mathcal{E}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

has the unique solution $\Delta_y^{(h)}(\mathcal{E}) = \Delta_z^{(h)}(\mathcal{E}) = 0$. The cases $\det(\mathcal{T}_2(\mathcal{E})) \neq 0$ and $\det(\mathcal{T}_3(\mathcal{E})) \neq 0$ are similar.

□

We define the regularized function $\tilde{d}_h : \mathfrak{U}_h \to \mathbb{R}$ by giving a sign to $d_h$, restricted to $\mathfrak{U}_h$, according to the following rules:

Definition 4.

$$\tilde{d}_h := \begin{cases} \mathrm{sign}(S_1)\, d_h & \textit{where } S_1 \neq 0 \\ \mathrm{sign}(S_2)\, d_h & \textit{where } S_2 \neq 0 \\ \mathrm{sign}(S_3)\, d_h & \textit{where } S_3 \neq 0 \end{cases} . \tag{21}$$

Note that $\tilde{d}_h$ is well-defined as the relations of Lemma 2 hold.

Using $(iii)$ of Proposition 1 we can continuously extend the function $\tilde{d}_h$ to the larger domain

$$\tilde{\mathfrak{U}}_h = \mathfrak{U} \setminus \{\mathcal{W}(\mathcal{T}) = 0\}$$

by defining $\tilde{d}_h(\mathcal{E}) = 0$ for each $\mathcal{E} \in \{S_1 = S_2 = S_3 = 0\} \setminus \{\mathcal{W}(\mathcal{T}) = 0\}$. We shall still call $\tilde{d}_h$ this extended function.

In Section 3.4 we shall prove that $\tilde{d}_h : \tilde{\mathfrak{U}}_h \to \mathbb{R}$ is an analytic function on the extended definition domain $\tilde{\mathfrak{U}}_h$, and its partial derivatives at a configuration

$\mathcal{E} \in \tilde{\mathfrak{U}}_h$ are given by the formula

$$\frac{\partial \tilde{d}_h}{\partial \mathcal{E}_k}(\mathcal{E}) = \left\langle \hat{\tau}_3(\mathcal{E}), \frac{\partial \Delta}{\partial \mathcal{E}_k}(\mathcal{E}, V_h(\mathcal{E})) \right\rangle \qquad k = 1 \ldots 10 \ . \tag{22}$$

### 3.3 Tangent crossings and degenerate critical points

We study the relation between the occurrence of a tangent crossing and the degeneration of the Hessian matrix of the squared Keplerian distance function evaluated at the corresponding minimum point. In particular we prove the following

PROPOSITION 2. *Let $\bar{\mathcal{E}}$ be a two–orbit configuration with $d_h(\bar{\mathcal{E}}) = 0$ and let $V_h(\bar{\mathcal{E}})$ be the corresponding minimum point of $d^2(\bar{\mathcal{E}}, \cdot)$. Then $V_h(\bar{\mathcal{E}})$ is a tangent crossing point if and only if the Hessian matrix $\mathcal{H}_V(d^2)(\bar{\mathcal{E}}, V_h(\bar{\mathcal{E}}))$ is degenerate.*

*Proof.* For a given configuration $\mathcal{E}$ the Hessian matrix at the minimum point $V_h(\mathcal{E})$ is

$$\mathcal{H}_V(d^2)(\mathcal{E}, V_h(\mathcal{E})) = 2 \begin{pmatrix} \Pi_1(\mathcal{E}) + |\tau_1(\mathcal{E})|^2 & -\langle \tau_1(\mathcal{E}), \tau_2(\mathcal{E}) \rangle \\ & \\ -\langle \tau_1(\mathcal{E}), \tau_2(\mathcal{E}) \rangle & -\Pi_2(\mathcal{E}) + |\tau_2(\mathcal{E})|^2 \end{pmatrix}$$

where

$$\begin{aligned} \Pi_1(\mathcal{E}) &= \left\langle \frac{\partial^2 \mathcal{X}_1}{\partial v_1^2}(E_1, v_1^{(h)}(\mathcal{E})), \Delta_h(\mathcal{E}) \right\rangle ; \\ \Pi_2(\mathcal{E}) &= \left\langle \frac{\partial^2 \mathcal{X}_2}{\partial v_2^2}(E_2, v_2^{(h)}(\mathcal{E})), \Delta_h(\mathcal{E}) \right\rangle ; \end{aligned}$$

thus the determinant of the Hessian matrix is

$$\begin{aligned} &\det \mathcal{H}_V(d^2)(\mathcal{E}, V_h(\mathcal{E})) = \\ &\quad = 4 \left\{ \left( \Pi_1(\mathcal{E}) + |\tau_1(\mathcal{E})|^2 \right) \left( -\Pi_2(\mathcal{E}) + |\tau_2(\mathcal{E})|^2 \right) - |\langle \tau_1(\mathcal{E}), \tau_2(\mathcal{E}) \rangle|^2 \right\} \ . \end{aligned}$$

If $\bar{\mathcal{E}}$ is a tangent crossing configuration with tangent crossing point $V_h(\bar{\mathcal{E}})$, then $\Delta_h(\bar{\mathcal{E}}) = 0$ and $\left|\langle \tau_1(\bar{\mathcal{E}}), \tau_2(\bar{\mathcal{E}}) \rangle\right| = \left|\tau_1(\bar{\mathcal{E}})\right| \left|\tau_2(\bar{\mathcal{E}})\right|$, so that

$$\det \mathcal{H}_V(d^2)(\bar{\mathcal{E}}, V_h(\bar{\mathcal{E}})) = 0 \ .$$

On the other hand, if $V = V_h(\bar{\mathcal{E}})$ is a crossing point (that is $\Delta_h(\bar{\mathcal{E}}) = 0$) and $\mathcal{H}_V(d^2)(\bar{\mathcal{E}}, V_h(\bar{\mathcal{E}}))$ is degenerate, then $\left|\langle \tau_1(\bar{\mathcal{E}}), \tau_2(\bar{\mathcal{E}}) \rangle\right| = \left|\tau_1(\bar{\mathcal{E}})\right| \left|\tau_2(\bar{\mathcal{E}})\right|$, so that $\tau_1(\bar{\mathcal{E}})$ and $\tau_2(\bar{\mathcal{E}})$ are parallel.

□

REMARK: If $v_1, v_2$ are the arc length parameters on their respective orbits, then the matrix $\mathcal{H}_V(d^2)(\bar{\mathcal{E}},\cdot)$ is degenerate at a critical point $V = V_j(\bar{\mathcal{E}})$ of $d^2(\bar{\mathcal{E}},\cdot)$ (not necessarily corresponding to an orbit crossing) if and only if relation

$$-\kappa_1\kappa_2\langle\nu_1,\Delta_j\rangle\langle\nu_2,\Delta_j\rangle + \kappa_1\langle\nu_1,\Delta_j\rangle - \kappa_2\langle\nu_2,\Delta_j\rangle + 1 - |\langle\tau_1,\tau_2\rangle|^2 = 0 \qquad (23)$$

holds at $\mathcal{E} = \bar{\mathcal{E}}$, where $\kappa_1(\mathcal{E}), \kappa_2(\mathcal{E}) > 0$ are the *curvatures* of the orbits and $\nu_1(\mathcal{E}), \nu_2(\mathcal{E})$ the normal vectors at the critical point (see [7]). Note that in this case $|\tau_1(\mathcal{E})| = |\tau_2(\mathcal{E})| = 1$).

### 3.4 Regularity of the minimal distance maps $\tilde{d}_h$

We shall prove the following result:

PROPOSITION 3. *The continuous map $\mathcal{E} \mapsto \tilde{d}_h(\mathcal{E})$ is analytic in $\tilde{\mathfrak{U}}_h$ and relation (22) gives a formula to compute its partial derivatives.*

*Proof.* Let $\bar{\mathcal{E}} \in \tilde{\mathfrak{U}}_h$. If $\tilde{d}_h(\bar{\mathcal{E}}) \neq 0$ we can prove the existence of the partial derivatives $\partial\tilde{d}_h/\partial\mathcal{E}_k$ using relation

$$\frac{\partial d_h}{\partial\mathcal{E}_k}(\bar{\mathcal{E}}) = \left\langle \hat{\Delta}_h(\bar{\mathcal{E}}), \frac{\partial\Delta}{\partial\mathcal{E}_k}(\bar{\mathcal{E}}, V_h(\bar{\mathcal{E}}))\right\rangle ,$$

the linearity of the scalar product and the linearity of the partial derivatives. Relation (22) immediately follows for this case, and it gives the local analyticity of the partial derivatives.

On the other hand the scalar product in the right-hand side of (22) makes sense also if $\tilde{d}_h(\bar{\mathcal{E}}) = 0$, but does the derivatives of $\tilde{d}_h$ exist in such point? We shall prove that also in this case these derivatives do exist and relation (22) gives a formula to compute them.

Let $\bar{\mathcal{E}} \in \tilde{\mathfrak{U}}_h$ with $\tilde{d}_h(\bar{\mathcal{E}}) = 0$. If for a given $k$ the case (ii) of Lemma 1 holds, then we have $d_h(\bar{\mathcal{E}} + \eta\hat{e}_k) > 0$ for $|\eta| > 0$ small enough. By Lagrange's theorem we have

$$\begin{aligned}
\frac{\partial^+\tilde{d}_h}{\partial\mathcal{E}_k}(\bar{\mathcal{E}}) &= \lim_{\eta_1\to 0^+}\frac{\tilde{d}_h(\bar{\mathcal{E}}+\eta_1\hat{e}_k) - \tilde{d}_h(\bar{\mathcal{E}})}{\eta_1} = \lim_{\eta_1\to 0^+}\frac{\partial\tilde{d}_h}{\partial\mathcal{E}_k}(\xi_k(\eta_1)) = \\
&= \lim_{\eta_2\to 0^-}\frac{\partial\tilde{d}_h}{\partial\mathcal{E}_k}(\xi_k(\eta_2)) = \lim_{\eta_2\to 0^-}\frac{\tilde{d}_h(\bar{\mathcal{E}}+\eta_2\hat{e}_k) - \tilde{d}_h(\bar{\mathcal{E}})}{\eta_2} = \frac{\partial^-\tilde{d}_h}{\partial\mathcal{E}_k}(\bar{\mathcal{E}})
\end{aligned}$$

where $\xi_k(\eta_1)$, $\xi_k(\eta_2)$ are two points on the segments joining $\bar{\mathcal{E}}$ and $\bar{\mathcal{E}}+\eta_r\hat{e}_k$, with $r = 1, 2$. Thus the partial derivative exists and can be computed by relation (22).

If (i) of Lemma 1 holds we have

$$\lim_{\eta\to 0}\frac{\tilde{d}_h(\bar{\mathcal{E}}+\eta\hat{e}_k) - \tilde{d}_h(\bar{\mathcal{E}})}{\eta} = 0$$

and the $k$–th partial derivative of $\tilde{d}_h$ in $\bar{\mathcal{E}}$ exists, it can be computed by relation (22) and is zero.

The previous discussion holds for each $k = 1 \dots 10$, thus all the partial derivatives of $\tilde{d}_h$ exist and are locally analytic at each $\bar{\mathcal{E}} \in \tilde{\mathfrak{U}}_h$. We conclude that the map $\tilde{d}_h$ itself is analytic in $\tilde{\mathfrak{U}}_h$.

□

REMARK: The map $\mathcal{E} \mapsto \tilde{d}_h(\mathcal{E})$ is not defined at the configurations such that the vectors $\tau_1$ and $\tau_2$, tangent to the orbits at the minimum point, are parallel. The map $\tilde{d}_h$ cannot be extended even continuously to such a configuration $\bar{\mathcal{E}}$ if the value of the minimal distance $d_h(\bar{\mathcal{E}})$ is not zero; actually $d_h$ is defined and continuous at $\bar{\mathcal{E}}$, but in the definition of $\tilde{d}_h$ we have changed sign to $d_h$ in each neighborhood of its. Nevertheless it is possible to extend continuously $\tilde{d}_h$ to the configurations $\bar{\mathcal{E}}$ such that $\tau_1 \parallel \tau_2$ provided $d_h(\bar{\mathcal{E}}) = 0$, but such an extension cannot be continuously differentiable, as shown in Appendix 8.3.

# 4 The uncertainty of the minimal distance

The orbit determination of a celestial body consists in the computation of a complete set of 6 orbital elements $(E, v)$ (the 5 components of the vector $E$ determine the orbit configuration in a given reference frame and the scalar $v$ is a parameter along the orbit) using its observations on the celestial sphere (e.g. the values of the right ascension and declination). These observations are affected by errors due to different reasons.

Already in classical orbit determination methods these errors were taken into account: Gauss' method (see [8]) provides us with a *nominal orbit*, solution of a least squares fit, together with its uncertainty, that can be represented by the $6 \times 6$ *covariance matrix* $\Gamma_{(E,v)} = C^{-1}_{(E,v)}$, the inverse of the *normal matrix*

$$C_{(E,v)} = \left[\frac{\partial \Xi}{\partial (E,v)}\right]^t \left[\frac{\partial \Xi}{\partial (E,v)}\right] ,$$

where $\Xi$ is the vector of the observational residuals.

The normal matrix defines a quadratic form approximating the sum of the squares of the residuals (the *target function*, see [13]). The *confidence region*, consisting of the set of possible orbital elements and defined by the target function, can thus be approximated using the normal matrix. This approximation gives a *confidence ellipsoid* centered in the nominal orbit in the 6–dimensional orbital element space.

We note that the uncertainty of the orbit configuration can be expressed by the covariance sub–matrix $\Gamma_E$, consisting of the principal $5 \times 5$ block of the matrix $\Gamma_{(E,v)}$:

$$\Gamma_{(E,v)} = \begin{pmatrix} \Gamma_E & \vdots \\ \dots & \Gamma_v \end{pmatrix} .$$

## 4.1 Covariance of the minimal distance

Given a nominal two–orbit configuration $\bar{\mathcal{E}} = (\bar{E}_1, \bar{E}_2)$, each orbit being taken with its covariance matrix $\Gamma_{\bar{E}_1}, \Gamma_{\bar{E}_2}$, we compute the uncertainty of the values of $\tilde{d}_h(\bar{\mathcal{E}})$ by making the following assumptions:

i) we can approximate the target function with the quadratic function defined by the normal matrix, as explained in the previous section;

ii) we can approximate the map $\mathcal{E} \mapsto \tilde{d}_h(\mathcal{E})$ with its linearization around the nominal configuration $\bar{\mathcal{E}}$;

iii) the determination of the two orbits are independent.

We expect that the hypotheses i), ii) give reliable results if the observations used in the orbit determination process do not give rise to a large confidence region. The hypothesis iii) is quite reasonable if either one of the bodies has a well determined orbit (e.g. a Solar system planet) or if the two sets of observations of the two bodies are independent. Using iii) the uncertainty of the two–orbit configuration $\bar{\mathcal{E}}$ can be expressed by the $10 \times 10$ covariance matrix

$$\Gamma_{\bar{\mathcal{E}}} = \left[ \begin{array}{cc} \Gamma_{\bar{E}_1} & 0 \\ 0 & \Gamma_{\bar{E}_2} \end{array} \right] .$$

We compute the covariance of $\tilde{d}_h(\bar{\mathcal{E}})$ by performing a linear propagation of the matrix $\Gamma_{\bar{\mathcal{E}}}$ (i.e. using assumption *ii)*):

$$\Gamma_{\tilde{d}_h(\bar{\mathcal{E}})} = \left[ \frac{\partial \tilde{d}_h}{\partial \mathcal{E}}(\bar{\mathcal{E}}) \right] \Gamma_{\bar{\mathcal{E}}} \left[ \frac{\partial \tilde{d}_h}{\partial \mathcal{E}}(\bar{\mathcal{E}}) \right]^t . \tag{24}$$

The *standard deviation*, defined as

$$\sigma_h(\bar{\mathcal{E}}) = \sqrt{\Gamma_{\tilde{d}_h(\bar{\mathcal{E}})}},$$

gives us a way to define a range of uncertainty for $\tilde{d}_h(\bar{\mathcal{E}})$: if we assume that the minimal distance $\tilde{d}_h(\bar{\mathcal{E}})$ is a Gaussian random variable (see [11]), there is a high probability ($\sim 99.7\%$) that its value is within the interval

$$I_h(\bar{\mathcal{E}}) = [\tilde{d}_h(\bar{\mathcal{E}}) - 3\sigma_h(\bar{\mathcal{E}}), \tilde{d}_h(\bar{\mathcal{E}}) + 3\sigma_h(\bar{\mathcal{E}})] . \tag{25}$$

Note that the regularity property of the map $\mathcal{E} \mapsto \tilde{d}_h(\mathcal{E})$ allows to use the covariance propagation formula (24) also for a vanishing distance. Furthermore the intervals (25) give meaningful possible values for the minimal distance $\tilde{d}_h(\bar{\mathcal{E}})$ because the latter may assume negative values.

### 4.2 Selection of a minimal distance map

The regularization of the minimal distance maps $d_h$ allows to eliminate the problems arising when they vanish; nevertheless this technique produces a singularity for the configurations such that the tangent vectors $\tau_1, \tau_2$ related to the corresponding minimum points are parallel.

Furthermore both families of maps, $d_h$ and the regularized maps $\tilde{d}_h$, present the same problems at the points $(\bar{\mathcal{E}}, V_h(\bar{\mathcal{E}}))$ where the Hessian matrix $\mathcal{H}_V(d^2)$ is degenerate.

In conclusion, when we want to estimate the uncertainty of the orbit distance between two Keplerian orbits, we should try to forecast if one of the problems previously illustrated can actually occur and, according to the results, we should select to use either the maps $d_h$ or the regularized maps $\tilde{d}_h$.

Using again a propagation formula, analogous to (24), we are able to compute the following quantities, useful to study the occurrence of these problems:

a) the uncertainty of the size of the angle between $\tau_1, \tau_2$, given by $|\hat{\tau}_1 \wedge \hat{\tau}_2|$, for each minimum point;

b) the uncertainty of $\det \mathcal{H}_V(d^2)(\bar{\mathcal{E}}, \cdot)$ at *all* the critical points[8];

c) the uncertainty of the mutual node distances $d^+_{nod}, d^-_{nod}$ at the ascending and descending mutual node (see (34)); if the uncertainty of the mutual orbital inclination $I_M$ (that we can compute as well) is such that $I_M$ cannot vanish, then $d_{min}$ is zero (hence one of the $d_h$ vanishes) if and only if one of the mutual node distances is zero. In this way we can control if there can be intersections between the orbits.

## 5 Applications to near–Earth asteroids

The near–Earth asteroids (NEAs) are the small bodies of the Solar system with a perihelion distance $q < 1.3$ AU.[9] During their secular evolution the Keplerian orbits of these asteroids precess and are subject to deformations, so that they can cross the orbit of the Earth and become dangerous for our planet.

Up to this date we know almost 3,800 NEAs[10]. When a new NEA is discovered the uncertainty of its Keplerian orbit may allow an intersection with the orbit of the Earth, hence it is particularly interesting to estimate the orbit distance for these objects.

In Table 1 we show the results of the computation of the uncertainty of the minimal distances for the asteroid Apophis (99942) and the orbit of the Earth. This NEA has been discovered on June 19, 2004 and has been observed only for two nights; it has been recovered in December 18, 2004 and then carefully followed up by the astronomers because the orbit uncertainty allowed a high probability impact with the Earth in 2029. Later on additional observations,

[8] note that even a saddle point could bifurcate into two saddle points and one minimum

[9] 1 AU (Astronomical Unit) ≈ 149,597,870 Km

[10] from the Near Earth Objects Dynamic Site (`http://newton.dm.unipi.it/neodys`)

including a few precovery observations of March 15, 2004, allowed to exclude this impact; nevertheless this asteroid is still followed up with interest because possible resonant returns due to close approaches with the Earth (see [18]) could lead to an impact in 2036.

| Name | $h$ | $\tilde{d}_h$ | $\tilde{d}_h - 3\sigma_h$ | $\tilde{d}_h + 3\sigma_h$ |
|---|---|---|---|---|
| $(99942)_a$ | 1 | 2.8378E-05 | -2.3328E-05 | 8.0084E-05 |
| | 2 | -5.1896E-02 | -5.1934E-02 | -5.1859E-02 |
| $(99942)_b$ | 1 | 4.4323E-05 | 4.0969E-05 | 4.7678E-05 |
| | 2 | -5.1885E-02 | -5.1887E-02 | -5.1883E-02 |

Table 1: The orbit $(99942)_a$ is computed with the set of observations of Apophis (99942) from the discovery (June 19, 2004) to December 24, 2004. The orbit $(99942)_b$ is computed by adding the precovery observations of March 15, 2004 to the previous set. For each of the two orbits we have 2 local minima of the Keplerian distance: for the first orbit $(99942)_a$ the uncertainty of the absolute minimum allows crossings with the orbit of the Earth.

We have computed two orbits with their uncertainty for Apophis (99942) by using the orbit determination software OrbFit[11]. The first orbit, $(99942)_a$ in the table, has been obtained with the observations from the discovery (June 19, 2004) to December 24, 2004; we have added the precovery observations of March 15, 2004 to obtain the second orbit, indicated in the table with $(99942)_b$.

In this example we have assumed, for simplicity, that the orbital elements of the Earth have zero uncertainty; in any case the contribution from the uncertainty of the Earth trajectory is negligible.

The uncertainty of the first orbit allows a crossing with the orbit of the Earth as the first local minimum $\tilde{d}_1$ can attain the values of the interval [-2.3328E-05, 8.0084E-05] (in AU). On the other hand the orbit distance uncertainty of the second orbit is such that crossings are excluded.

# 6 Conclusions and further work

The regularization of the minimal distance maps introduced in this paper allows to define a meaningful uncertainty of the orbit distance between two Keplerian orbits even if this uncertainty leads to negative values of the distance. Moreover the orbit crossing singularity is removed, except for the tangent crossing case.

The reliability of the linearity assumptions done in Section 4.1 still needs to be investigated. We expect that if the orbit of a recently discovered asteroid is poorly determined, then these hypotheses could fail, and the results of the computation of the intervals $I_h$ defined in (25) might be inaccurate.

We plan also to make extensive numerical experiments with a large database of known asteroids, like the one of the web site ASTDyS[12]. The classification of the asteroids based on the nominal orbit distance can change by taking into

[11]available at the web address `http://newton.dm.unipi.it/orbfit`

[12]the Asteroid Dynamic Site (`http://hamilton.dm.unipi.it/astdys`)

account the orbit uncertainty and we would like to investigate this feature with particular care of possible small values of the orbit distance with the Earth.

# 7 Acknowledgements

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# 8 Appendix

## 8.1 Maximal number of orbit intersections

We prove the following result:

PROPOSITION 4. *Two Keplerian orbits with a common focus cannot intersect each other in more than two distinct points. Furthermore if the two conics are coplanar, then there is only one orbit crossing if and only if it is a tangent crossing.*

*Proof.* If there were more than two intersections, then the two orbits would be coplanar. Let us consider a parameterization of the two orbits using the true anomalies $v_1, v_2$ and a reference plane in which the $x$ axis is directed towards the pericenter of the first orbit:

$$\left\{ \begin{array}{ll} x_1 & = r_1 \cos v_1 \\ y_1 & = r_1 \sin v_1 \end{array} \right. \qquad \left\{ \begin{array}{ll} x_2 & = r_2 \left[\cos\omega \cos v_2 - \sin\omega \sin v_2\right] \\ y_2 & = r_2 \left[\sin\omega \cos v_2 + \cos\omega \sin v_2\right] \end{array} \right.$$

with

$$r_1 = \frac{q_1(1+e_1)}{1+e_1\cos v_1}\,; \qquad r_2 = \frac{q_2(1+e_2)}{1+e_2\cos v_2}\,;$$

$q_1, q_2$ the two pericenter distances, $e_1, e_2$ the orbit eccentricities and $\omega$ the angle between the pericenter of the second orbit and the $x$ axis (that is the angular difference between the two pericenters).

By setting $r_1 = r_2$ and $x_1 = x_2$ we obtain the two trigonometric equations

$$\begin{array}{l} q_1(1+e_1)(1+e_2\cos v_2) = q_2(1+e_2)(1+e_1\cos v_1)\,; \\ \cos v_1 = \cos\omega\cos v_2 - \sin\omega \sin v_2\,; \end{array}$$

from which we have

$$C_1 \cos v_2 + C_2 \sin v_2 = C_3 \tag{26}$$

with

$$\begin{array}{rcl} C_1 & = & q_1(1+e_1)e_2 - q_2(1+e_2)e_1\cos\omega\,; \\ C_2 & = & q_2(1+e_2)e_1 \sin\omega\,; \\ C_3 & = & q_2(1+e_2) - q_1(1+e_1)\,; \end{array}$$

which gives at most two solutions for the $v_2$ variable. From the crossing relations $x_1 = x_2$ and $y_1 = y_2$ we easily see that there is at most one real value of $v_1$ corresponding to a given value of $v_2$ that satisfies both equations.

Using the coordinate change defined by $t = \tan(v_2/2)$ we find the second degree equation in the variable $t$

$$(C_1 + C_3)t^2 - 2C_2 t + (C_3 - C_1) = 0\,, \tag{27}$$

giving the solutions of (26): its roots are

$$t_{1,2} = \frac{C_2 \pm \sqrt{C_2^2 - C_3^2 + C_1^2}}{C_1 + C_3}\ . \tag{28}$$

From this relation we can deduce the tangent crossing condition, corresponding to the vanishing of the discriminant $C_2^2 - C_3^2 + C_1^2$:

$$q_1^2(1+e_1)^2(e_2^2-1)+q_2^2(1+e_2)^2(e_1^2-1)+2q_1q_2(1+e_1)(1+e_2)(1-e_1e_2\cos\omega) = 0\,.$$

□

Remark: For two elliptical orbits with semimajor axis $a_1, a_2$ the tangent crossing condition can be written as

$$a_1^2(1-e_1^2) + a_2^2(1-e_2^2) - 2a_1a_2(1-e_1e_2\cos\omega) = 0\ . \tag{29}$$

The coordinate change that we have selected does not allow to find the value $v_2 = \pi$, that is sent to infinity: note that $v_2 = \pi$ is a solution if and only if $C_1 + C_3 = 0$, that is

$$q_2(1+e_2)(1-e_1\cos\omega) - q_1(1+e_1)(1-e_2) = 0\,. \tag{30}$$

If (30) holds, and $v_2 = \pi$ is a solution, then (27) becomes a linear equation that can give only one additional value of $v_2$, solution of (26).

## 8.2 The mutual elements

Let us consider two Keplerian orbits with a common focus, whose configurations are represented by the *cometary elements*

$$E_1 = (q_1, e_1, I_1, \Omega_1, \omega_1)\,; \qquad E_2 = (q_2, e_2, I_2, \Omega_2, \omega_2) \tag{31}$$

in a given reference frame, that are respectively the pericenter distance, the eccentricity, the inclination, the longitude of the ascending node and the pericenter argument.

If the two orbits are not coplanar we can define the *mutual nodal line* as the intersection of the orbital plane; we also define as *mutual node* each pair of points on the mutual nodal line, one belonging to the first orbit and the other to the second, that lie on the same side of the mutual nodal line with respect to the common focus.

By assigning an orientation to both orbits, i.e. a normal vector $\mathbf{N}_i$ $(i = 1, 2)$ to each orbital plane we have an *ascending* and a *descending* mutual node of the second orbit with respect to the first one (see Figure 4). We define the *cometary mutual elements*

$$\{q_1, e_1, q_2, e_2, I_M, \omega_M^{(1)}, \omega_M^{(2)}\}, \tag{32}$$

where $I_M$ is the *mutual inclination*, that is the angle between $\mathbf{N}_1$ and $\mathbf{N}_2$, and $\omega_M^{(1)}, \omega_M^{(2)}$ are the angles between the ascending mutual node and the pericenters of the two orbits.

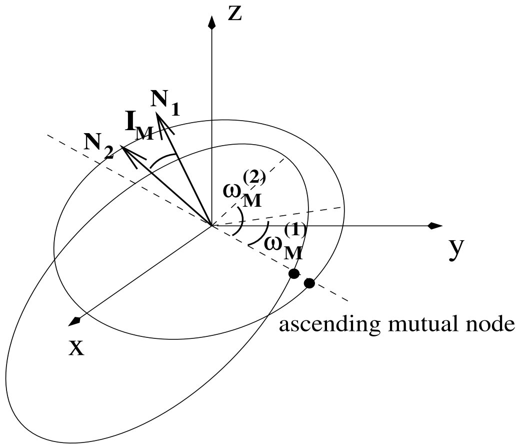


Figure 4: We draw some of the mutual elements of two orbits, together with the orientation vectors $\mathbf{N}_1, \mathbf{N}_2$, that define the mutual inclination $I_M$ and the ascending mutual node.

We can express $I_M$, $\omega_M^{(1)}$, $\omega_M^{(2)}$ as functions of the cometary elements: first we note that we can select

$$\mathbf{N}_1 = \begin{pmatrix} \sin\Omega_1 \sin I_1 \\ -\cos\Omega_1 \sin I_1 \\ \cos I_1 \end{pmatrix}; \qquad \mathbf{N}_2 = \begin{pmatrix} \sin\Omega_2 \sin I_2 \\ -\cos\Omega_2 \sin I_2 \\ \cos I_2 \end{pmatrix};$$

then the mutual inclination $I_M \in [0, \pi[$ is defined by $\cos I_M = \langle \mathbf{N}_1 \, , \mathbf{N}_2 \rangle$.

The unit vectors

$$\mathcal{A}_{nod} = \frac{\mathbf{N}_1 \wedge \mathbf{N}_2}{|\mathbf{N}_1 \wedge \mathbf{N}_2|} \tag{33}$$

(pointing to the ascending mutual node),

$$\mathcal{X}_1 = \begin{pmatrix} \cos\Omega_1 \cos\omega_1 - \sin\Omega_1 \sin\omega_1 \cos I_1 \\ \sin\Omega_1 \cos\omega_1 + \cos\Omega_1 \sin\omega_1 \cos I_1 \\ \sin\omega_1 \sin I_1 \end{pmatrix};$$

$$\mathcal{X}_2 = \begin{pmatrix} \cos\Omega_2 \cos\omega_2 - \sin\Omega_2 \sin\omega_2 \cos I_2 \\ \sin\Omega_2 \cos\omega_2 + \cos\Omega_2 \sin\omega_2 \cos I_2 \\ \sin\omega_2 \sin I_2 \end{pmatrix}$$

(pointing to the positions of the bodies) give us the mutual pericenter arguments

$$\cos\omega_M^{(1)} = \langle \mathcal{A}_{nod}\,, \mathcal{X}_1 \rangle\,; \qquad \sin\omega_M^{(1)} = \langle \mathcal{A}_{nod} \times \mathcal{X}_1, \mathbf{N}_1 \rangle\,;$$
$$\cos\omega_M^{(2)} = \langle \mathcal{A}_{nod}\,, \mathcal{X}_2 \rangle\,; \qquad \sin\omega_M^{(2)} = \langle \mathcal{A}_{nod} \times \mathcal{X}_2, \mathbf{N}_2 \rangle\ .$$

The ascending and descending mutual node distances are respectively defined by

$$\begin{aligned} d_{nod}^{+} &= \frac{q_1(1+e_1)}{1+e_1\cos\omega_M^{(1)}} - \frac{q_2(1+e_2)}{1+e_2\cos\omega_M^{(2)}}\,; \\ d_{nod}^{-} &= \frac{q_1(1+e_1)}{1-e_1\cos\omega_M^{(1)}} - \frac{q_2(1+e_2)}{1-e_2\cos\omega_M^{(2)}}\ . \end{aligned} \qquad (34)$$

## 8.3 Lack of regularity for $\tau_1 \parallel \tau_2$

In this section we prove that some partial derivatives of the regularized map $\mathcal{E} \mapsto \tilde{d}_h(\mathcal{E})$ cannot be continuously extended to the tangent crossing configurations $\bar{\mathcal{E}}$, even if we can obtain a continuous extension of the map itself to these points by setting $\tilde{d}_h(\bar{\mathcal{E}}) = 0$. In particular we shall prove that there are two paths of configurations, $\mathcal{E}^\alpha(\epsilon)$ and $\mathcal{E}^\beta(\epsilon)$ with $\epsilon \in [0, \bar{\epsilon}]$, such that

$$\lim_{\epsilon\to 0^+} \mathcal{E}^\alpha(\epsilon) = \lim_{\epsilon\to 0^+} \mathcal{E}^\beta(\epsilon) = \bar{\mathcal{E}}$$

and

$$\lim_{\epsilon\to 0^+} \frac{\partial \tilde{d}_h}{\partial \mathcal{E}_k}(\mathcal{E}^\alpha(\epsilon), V_h(\mathcal{E}^\alpha(\epsilon))) \neq \lim_{\epsilon\to 0^+} \frac{\partial \tilde{d}_h}{\partial \mathcal{E}_k}(\mathcal{E}^\beta(\epsilon), V_h(\mathcal{E}^\beta(\epsilon)))$$

for some element $\mathcal{E}_k$.

This implies that $\tilde{d}_h$ cannot be continuously differentiable at tangent crossing configurations.

We shall take into account the case of two elliptic orbits, but the same proof may be adapted to the case of different conics.

Let us use the set of cometary elements, that is the configuration elements (31) and the true anomalies $v_1$, $v_2$ as parameters along the orbits. We can choose the path $\mathcal{E}^\alpha(\epsilon)$ in a way that only the pericenter distance of one orbit changes with $\epsilon$ and we have $d_h(\mathcal{E}^\alpha(\epsilon)) = \tilde{d}_h(\mathcal{E}^\alpha(\epsilon)) = 0$ for each $\epsilon$. Thus we obtain

$$\lim_{\epsilon\to 0^+} \frac{\partial \tilde{d}_h}{\partial \mathcal{E}_k}(\mathcal{E}^\alpha(\epsilon), V_h(\mathcal{E}^\alpha(\epsilon))) = 0\ .$$

Let us select the second path $\mathcal{E}^\beta(\epsilon)$. Without loss of generality we can choose a reference frame so that the Cartesian coordinates of the first orbit are

$$x_1 = r_1\cos v_1\,; \qquad y_1 = r_1 \sin v_1\,; \qquad z_1 = 0$$

with

$$r_1 = \frac{q_1(1+e_1)}{1+e_1\cos v_1}\,,$$

where $q_1$ is the pericenter distance, $e_1$ the eccentricity and $v_1$ the true anomaly.

In the orbit plane we consider the straight line $L$ passing through the tangent crossing point $P$, with the direction of the outgoing[13] normal $\hat{n}$ to both conics (see Figure 5). For each $\epsilon > 0$ we select a point $P_\epsilon$ moving continuously and monotonically with $\epsilon$ on the line $L$ and such that $\lim_{\epsilon\to 0} P_\epsilon = P$. Furthermore we consider the family of vectors $\tau_{2,\epsilon}$, each obtained by a counter–clockwise rotation of the tangent vector $\tau_2$ of the second orbit at the crossing point around the axis $\hat{n}$ by an angle $\epsilon$. Note that $\tau_{2,0} = \tau_2$.

We define the path of orbits $\mathcal{E}^\beta(\epsilon)$ by changing only the second ellipse with $\epsilon$, in a way that for each $\epsilon$ the second orbit shares the same focus with the first, it passes through $P_\epsilon$ and has $\tau_{2,\epsilon}$ as tangent vector in $P_\epsilon$ (see Figure 5).

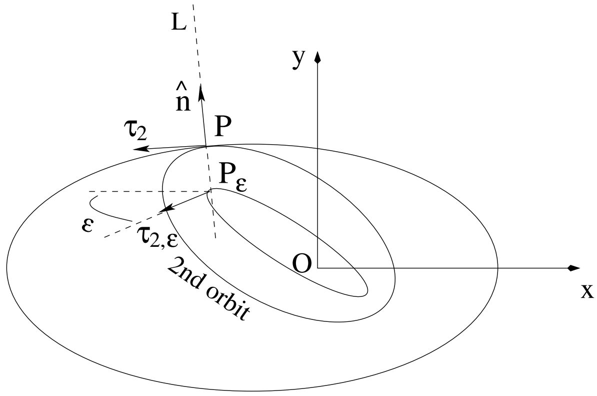


Figure 5: Sketch of the geometric construction to select the orbit path $\mathcal{E}^\beta(\epsilon)$.

With this choice of $\mathcal{E}^\beta(\epsilon)$ the derivatives of $\tilde{d}_h$ with respect to $q_1$ are given by the formula

$$\frac{\partial \tilde{d}_h}{\partial q_1}(\mathcal{E}^\beta(\epsilon)) = \left\langle \hat{n}, \frac{\partial \Delta}{\partial q_1}(\mathcal{E}^\beta(\epsilon), V_h(\mathcal{E}^\beta(\epsilon))) \right\rangle$$

where $\hat{n}$ does not depend on $\epsilon$, so that

$$\lim_{\epsilon\to 0} \frac{\partial \tilde{d}_h}{\partial q_1}(\mathcal{E}^\beta(\epsilon)) = \left\langle \hat{n}, \frac{\partial \Delta}{\partial q_1}(\bar{\mathcal{E}}, V_h(\bar{\mathcal{E}})) \right\rangle$$

due to the continuity of $\frac{\partial \Delta}{\partial q_1}$.

Let $v_* = v_1^{(h)}(\bar{\mathcal{E}})$ be the value of the true anomaly on the first orbit corresponding to the tangent crossing. The components of the vector $\hat{n} = (\alpha, \beta, 0)$ must satisfy

$$\begin{aligned} -\alpha \sin v_* + \beta(\cos v_* + e_1) &= 0\,; \\ \alpha^2 + \beta^2 &= 1\,; \end{aligned}$$

[13]with respect to the convex area enclosed

thus, if $\sin v_* = 0$ then $\alpha = 1$, $\beta = 0$, otherwise

$$\alpha = \frac{(\cos v_* + e_1)}{\sqrt{1 + 2\, e_1\, \cos v_* + e_1^2}}\,; \qquad \beta = \frac{\sin v_*}{\sqrt{1 + 2\, e_1\, \cos v_* + e_1^2}}\ .$$

From the relation

$$\left\langle \hat{n}, \frac{\partial \Delta}{\partial q_1} \right\rangle = \alpha\, \frac{\partial x_1}{\partial q_1} + \beta\, \frac{\partial y_1}{\partial q_1}$$

we obtain

$$\left\langle \hat{n}, \frac{\partial \Delta}{\partial q_1}(\bar{\mathcal{E}}, V_h(\bar{\mathcal{E}})) \right\rangle = \frac{1 + e_1}{\sqrt{1 + 2\, e_1\, \cos v_* + e_1^2}} \neq 0\ .$$

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