Periodic orbits of the N-body problem with the symmetry of Platonic polyhedra

Giovanni Federico Gronchi

Abstract We review some recently discovered periodic orbits of the N-body problem [8], whose existence is proved by means of variational methods. These orbits are minimizers of the Lagrangian action functional in a set of T-periodic loops, equivariant for the action of a group G and satisfying some topological constraints. Both the group action and the topological constraints are defined using the symmetry of Platonic polyhedra.

1 Introduction

The N-body problem is the study of the motion of N particles, regarded as point masses, which are subject to their mutual gravitational interaction. This problem has been investigated for a long time and is useful for different purposes, e.g. to study the stability of the solar system planets, or to predict possible collisions of asteroids with the Earth.
For $N \geq 3$ the problem is not integrable (in the sense of classical Mechanics) and, even worse, chaoticity phenomena may prevent a reliable computation of the solutions with numerical methods over the desired time span.

The importance of periodic orbits in the study of the N-body problem is well expressed by the words of Poincaré: *'ce qui rend ces solutions périodiques aussi précieuses, c'est qu'elle sont, pour ainsi dire, la seule brèche par où nous puissions pénétrer dans une place jusqu'ici réputée inabordable'* [13].
In the literature there are different ways to prove the existence of periodic solutions of this problem: continuation of known solutions, expansions by series, topological methods, etc. Here we consider solutions whose existence is proved by means of Calculus of Variations. In the last years some new

G. F. Gronchi
Dipartimento di Matematica, Università di Pisa, Largo B. Pontecorvo, 5, 56127 Pisa (Italy), e-mail: gronchi@dm.unipi.it

and unexpected periodic orbits of the N-body problem have been found by variational methods; one of these is the 'figure eight' [14], [4], where three equal masses follow the same planar, eight-shaped curve, with the same time law and a shift of 1/3 of the period one after the other. A motion of this kind is called *choreography* [17]. For other results on periodic orbits of the N-body problem see [2], [7], [8] and references therein.
In this short paper we review some of the results in [8], where the symmetry groups of Platonic polyhedra play a fundamental role.

2 Calculus of Variations for periodic orbits

We can choose the units so that the equations of motion of the N-body problem are

$$m_i \ddot{u}_i = U_{u_i}(u), \quad i = 1, \dots, N, \tag{1}$$

where $m_1, \dots, m_N$ are the masses of the particles and

$$U(u) = \sum_{1 \le h < k \le N} \frac{m_h m_k}{|u_h - u_k|}$$

is the potential. The map $\mathbb{R} \ni t \to u = (u_1, \dots, u_N) \in \mathbb{R}^{3N}$ describes the evolution of the positions of the N bodies in $\mathbb{R}^3$.
We introduce the kinetic energy

$$K(\dot{u}) = \frac{1}{2} \sum_{i=1}^{N} m_i |\dot{u}_i|^2.$$

For a fixed period $T > 0$ we can give a variational formulation to the search for periodic orbits of (1): the solutions are stationary points of the Lagrangian action functional

$$\mathcal{A}(u) = \int_0^T K(\dot{u}(t)) + U(u(t))\, dt \tag{2}$$

on a set of admissible curves. In particular, we can search for periodic solutions which are minimum points of $\mathcal{A}$.
A natural environment for this problem is the Sobolev space $H^1_T(\mathbb{R}, \mathbb{R}^{3N})$ of T-periodic *loops* (i.e. closed curves) in H^1, with norm

$$\|u\|_{H^1_T} = \Big(\int_0^T |u(t)|^2\, dt + \int_0^T |\dot{u}(t)|^2\, dt \Big)^{1/2},$$

where $\dot{u} \in L^2$ is the weak derivative of u in H^1.
Using the integrals of the center of mass we can assume $\sum_{h=1}^N m_h u_h = 0$. We introduce the configuration space

$$\mathcal{X} = \Big\{ x = (x_1, \dots, x_N) \in \mathbb{R}^{3N} : \sum_{h=1}^{N} m_h x_h = 0 \Big\}$$

and consider the loop space $\Lambda = H^1_T(\mathbb{R}, \mathcal{X})$.

Let $E \subseteq \Lambda$ be a set of loops. We say that a functional $\mathcal{J} : \Lambda \to \mathbb{R}$ is *coercive* on E if $\lim_{k\to\infty} \mathcal{J}(u^{(k)}) = +\infty$ for each sequence $\{u^{(k)}\}_{k\in\mathbb{N}} \subset E$ such that $\lim_{k\to\infty} \|u^{(k)}\|_{H^1_T} = +\infty$. Standard methods of Calculus of Variations, going back to Tonelli [18], ensure the existence of minimum points of $\mathcal{J}$ in $\overline{E}$ (the closure of E in Λ) provided $\mathcal{J}$ is coercive on E.
However, the action functional $\mathcal{A}$ is not coercive on the whole space Λ; we can even find sequences $\{u^{(k)}\}_{k\in\mathbb{N}} \subset \Lambda$ such that

$$\lim_{k\to\infty} \|u^{(k)}\|_{H^1_T} = +\infty \qquad \text{and} \qquad \lim_{k\to\infty} \mathcal{A}(u^{(k)}) = 0.$$

Another obstruction is that minimizers of $\mathcal{A}$ may have collisions: this is a consequence of Sundman's estimates [20], [1], and had already been noted by Poincaré [16] for the case of three bodies. In fact, if u_* is a minimizer with collisions, then close to a collision time t_c we have

$$|u_*(t)| = O(|t_c - t|^{2/3}),$$

therefore the contribution of collisions to the action $\mathcal{A}$ is finite.

We can recover coercivity by introducing constraints to the admissible loops. For example in [10] the author proves that the whole family of isochronous Keplerian orbits, including degenerate collision-ejection solutions, minimizes the Lagrangian action of the Kepler problem in the set of T-periodic loops in $H^1(\mathbb{R}, \mathbb{R}^2)$ winding around the center of attraction exactly once, either clockwise or counter-clockwise. This is a *topological constraint* and is sufficient to have coercivity.
Another way to obtain coercivity is by *symmetry constraints*. For example in [6], [5] the authors restrict the minimization of the action to the maps $u(t) \in \Lambda$ such that

$$u(t + T/2) = -u(t).$$

In both these examples the trajectories of any sequence $\{u^{(k)}\}_{k\in\mathbb{N}}$ with $\lim_{k\to\infty} \|u^{(k)}\|_{H^1_T} = +\infty$ is such that

$$\lim_{k\to\infty} \max_{s,t\in\mathbb{R}} |u^{(k)}(s) - u^{(k)}(t)| = +\infty,$$

so that the kinetic part of the action diverges:

$$\lim_{k\to\infty} \int_0^T K(\dot{u}^{(k)}(t))\, dt = +\infty.$$

A general formulation for the use of symmetry constraints to prove the existence of periodic orbits is given in [7]. Here the authors restrict the admissible curves to the set Λ_G of loops equivariant with respect to the action of a finite group G on the space $\Lambda = H^1_T(\mathbb{R}, \mathcal{X})$.
Assume $m_i = m_j$ for $1 \le i < j \le N$. Consider a finite group G and its three representations

$$\tau : G \to O(2), \qquad \rho : G \to O(3), \qquad \sigma : G \to \Sigma_N,$$

that are homomorphisms on the orthogonal groups $O(2)$, $O(3)$ and on the symmetric group Σ_N of permutations of N elements. Through τ, ρ, σ we can define an action of G on the loop space Λ by

$$G \times \Lambda \ni (g, u) \mapsto g \cdot u \in \Lambda, \qquad (g \cdot u)_j(t) := \rho(g) u_{\sigma(g^{-1})(j)}(\tau(g^{-1})t)$$

and an action on the configuration space $\mathcal{X}$ by

$$G \times \mathcal{X} \ni (g, x) \mapsto g \cdot x \in \mathcal{X}, \qquad (g \cdot x)_j := \rho(g) x_{\sigma(g^{-1})(j)},$$

for each $j = 1 \dots N$, $t \in \mathbb{R}$. We denote by Λ_G the set of G-equivariant loops in Λ: $u \in \Lambda$ belongs to Λ_G if and only if

$$g \cdot u(t) = u(t), \qquad \forall\, g \in G,\ \forall t \in \mathbb{R}.$$

Moreover, let $\mathcal{X}^G$ be the set of configurations in $\mathcal{X}$ which are fixed by every element of G. A necessary and sufficient condition for coercivity (see [7]) is given by

Proposition 1. *The Lagrangian action $\mathcal{A}$, restricted to the set of G-equivariant loops Λ_G, is coercive if and only if $\mathcal{X}^G = \{0\}$.*

Since $\mathcal{X}^G = \{0\}$ if and only if

$$\frac{1}{T}\int_0^T u_j(t)\, dt = 0, \qquad j = 1, .., N, \tag{3}$$

the trajectories of the N particles must share a common center (in the sense of the integral average). This is a strong constraint to the motion.

We discuss now the problem of collisions. In the literature there are different methods to show that a minimizer of $\mathcal{A}$ is *collision-free*. We consider two classes of methods:

i) **level estimates**: compute an *a priori* lower bound a for $\mathcal{A}(u_*)$, where u_* is any minimizer with collisions, and find an admissible loop v such that

$$\mathcal{A}(v) < a \le \mathcal{A}(u_*);$$

ii) **local perturbations**: for each minimizer u_* with collisions find a small admissible perturbation $u_* + \xi$ such that

$$\mathcal{A}(u_* + \xi) < \mathcal{A}(u_*).$$

We recall two results that are useful to define local perturbations. The first is due to C. Marchal [3]. In the Kepler problem with center of force O the action of the parabolic collision-ejection arc AOB (going from A to B through O) is greater than the action of the isochronous direct Keplerian arc joining A with B and, provided $A \neq B$, it is also greater than the action of the indirect Keplerian arc. If $A = B$ the indirect arc does not exist. We can construct local perturbations of collision solutions using these Keplerian arcs combined with a blow–up technique (see [7]).
The second result is also based on an idea by Marchal [12], [3], and has been later generalized in [7]. Consider a family of admissible perturbations u^σ, parametrized by the directions σ spanning the whole unit sphere $\mathbb{S}^2$. If the integral average $\frac{1}{4\pi}\int_{\mathbb{S}^2} \mathcal{A}(u^\sigma)d\sigma$ of the values of the action over $\mathbb{S}^2$ is less than $\mathcal{A}(u_*)$, then there exists a perturbation with lower action than u_*, even if we cannot exhibit it explicitly.

3 Platonic polyhedra and orbits with symmetry and topological constraints

We present some of the results in [8], where new periodic orbits of the N-body problem with the symmetry of Platonic polyhedra have been introduced.
The symmetry of Platonic polyhedra was already associated to the motion of celestial bodies by J. Kepler at the end of the XVI century [11].

Let $\mathcal{R} \in \{\mathcal{T}, \mathcal{O}, \mathcal{I}\}$, where $\mathcal{T}, \mathcal{O}, \mathcal{I}$ are the rotation groups of tetrahedron, octahedron, icosahedron. Hexahedron and dodecahedron have the same rotation group as octahedron and icosahedron respectively. We set $N = |\mathcal{R}|$ and assume $m_i = m_j = 1$ for $i, j = 1 \ldots N$.
We restrict the Lagrangian action to open cones

$$\mathcal{K} = \{u \in H^1_T(\mathbb{R}, \mathcal{X}) : u \text{ satisfies (a), (b)}\}$$

where

(a) the motion u_j of the j-th particle is determined by the motion u_1 of the first particle (that we call *generating particle*) through the relation

$$u_j = R_j u_1, \qquad j \in \{1, \ldots, N\},$$

with $\{1, \ldots, N\} \ni j \to R_j \in \mathcal{R}$ a bijection such that $R_1 = I$;

(b) the motion u_1 of the generating particle belongs to a given free homotopy class in $\mathbb{R}^3 \setminus \Gamma$, where

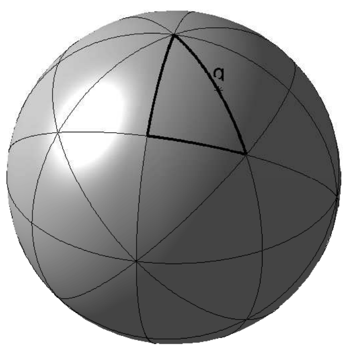
$$\Gamma = \cup_{R \in \mathcal{R} \setminus \{I\}} r(R)$$

with $r(R)$ the rotation axis of R.

Conditions (a), (b) are a symmetry and a topological constraint respectively. The superposition of these constraints allows us to produce orbits that do not share a common center, unlike the ones in [7]. Moreover, assuming condition (a) holds, the set Γ in condition (b) is exactly the set of points where collisions take place, if any.

3.1 Encoding the cones $\mathcal{K}$

We describe two ways to encode the topological constraints defining the cones $\mathcal{K}$. Let $\tilde{\mathcal{R}}$ be the full symmetry group (including reflections) related to $\mathcal{R}$. The reflection planes induce a tessellation of the unit sphere $\mathbb{S}^2$, as in Figure 1, with $2N$ spherical triangles. Each vertex of such triangles corresponds to a



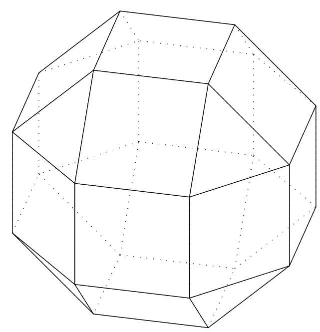


Fig. 1 Tessellation of $\mathbb{S}^2$ for $\mathcal{R} = \mathcal{O}$ and the Archimedean polyhedron $\mathcal{Q}_{\mathcal{O}}$.

pole $p \in \mathcal{P} = \Gamma \cap \mathbb{S}^2$. Select one of these triangles, say τ. By a suitable choice of a point $q \in \partial\tau$ (see Figure 1) we can define an Archimedean polyhedron $\mathcal{Q}_{\mathcal{R}}$, which is the convex hull of the orbit of q under $\mathcal{R}$, and therefore is strictly related with the symmetry group $\mathcal{R}$. For details see [8].
We can characterize a cone $\mathcal{K}$ by a periodic sequence $\mathfrak{t} = \{\tau_k\}_{k\in\mathbb{Z}}$ of triangles of the tessellation such that τ_{k+1} shares an edge with τ_k and $\tau_{k+1} \neq \tau_{k-1}$ for each $k \in \mathbb{Z}$. This sequence is uniquely determined by $\mathcal{K}$ up to translations, and describes the homotopy class of the admissible paths followed by the generating particle (see Figure 2, left).
We can also characterize $\mathcal{K}$ by a periodic sequence $\nu = \{\nu_k\}_{k\in\mathbb{Z}}$ of vertexes of $\mathcal{Q}_{\mathcal{R}}$ such that the segment $[\nu_k, \nu_{k+1}]$ is an edge of $\mathcal{Q}_{\mathcal{R}}$ and $\nu_{k+1} \neq \nu_{k-1}$ for each $k \in \mathbb{Z}$. Also the sequence ν is uniquely determined by $\mathcal{K}$ up to translations, and with it we can construct a piecewise linear loop, joining consecutive vertexes ν_k with constant speed, that represents a possible motion of the generating particle (see Figure 2, right).

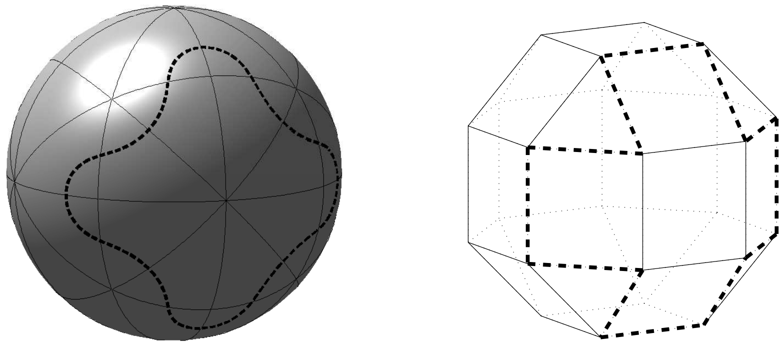


Fig. 2 Encoding a cone $\mathcal{K}$. Left: the dashed path on $\mathbb{S}^2$ describes the periodic sequence $\mathbf{t}$ of triangles of the tessellation. Right: the dashed piecewise linear path describes the corresponding periodic sequence ν of vertexes of $\mathcal{Q}_{\mathcal{O}}$.

3.2 Existence of minimizers

We consider cones $\mathcal{K} \subset \Lambda$ such that the trajectory of u_1 does not wind around one rotation axis only. In this case we can prove that $\mathcal{A}|_{\mathcal{K}}$ is coercive (see [8], Proposition 4.1).

If $\mathcal{A}$ is coercive, standard methods of Calculus of Variations yield the existence of a minimizer. Let $\{u^{(h)}\}_{h\in\mathbb{N}} \subset \mathcal{K}$ be a minimizing sequence, i.e. $\lim_{h\to\infty} \mathcal{A}(u^{(h)}) = \inf_{u\in\mathcal{K}} \mathcal{A}(u)$. Up to subsequences, we can find a constant $M > 0$ such that $\mathcal{A}(u^{(h)}) \leq M$ for each $h \in \mathbb{N}$. We can take M equal to the action of the piecewise linear loop defined by the sequence ν that characterizes K. Then coercivity implies

$$\|u^{(h)}\|_{H^1_T} \leq C, \qquad h \in \mathbb{N} \tag{4}$$

for a constant $C > 0$. Since $\Lambda = H^1_T(\mathbb{R}, \mathcal{X})$ is a reflexive space, there exists a subsequence of $\{u^{(h)}\}_{h\in\mathbb{N}}$ weakly converging to a loop u_* in Λ. From the bound (4) we also have

$$\|u^h\|_\infty \leq C, \qquad |u^{(h)}(t_1) - u^{(h)}(t_2)| \leq C|t_1 - t_2|^{1/2},$$

therefore, by the Ascoli-Arzelà theorem, there exists a subsequence of $\{u^{(h)}\}_{h\in\mathbb{N}}$ uniformly converging to u_* on compact sets. Hence $u_* \in \Lambda \cap \overline{\mathcal{K}}$, with $\overline{\mathcal{K}}$ the closure of $\mathcal{K}$ in the C^0 topology.

Moreover, the action functional $\mathcal{A}$ is weakly lower semi-continuous in Λ. In fact, if $\{u^{(h)}\}_{h\in\mathbb{N}}$ weakly converges to u_* in Λ, then it converges to u_* also uniformly (up to subsequences). By Cauchy-Schwartz inequality we have

$$\liminf_{h\to\infty} \|u^{(h)}\|^2_{H^1_T} \geq \|u_*\|^2_{H^1_T}. \tag{5}$$

Then relation (5) and the uniform convergence of $\{u^{(h)}\}_{h\in\mathbb{N}}$ imply

$$\liminf_{h\to\infty}\int_0^T |\dot{u}^{(h)}(t)|^2\,dt \geq \int_0^T |\dot{u}_*(t)|^2\,dt,$$

and

$$\lim_{h\to\infty}\int_0^T U(u^{(h)}(t))\,dt = \int_0^T U(u_*(t))\,dt,$$

so that

$$\liminf_{h\to\infty}\mathcal{A}(u^{(h)}) \geq \mathcal{A}(u_*),$$

If $\{u^{(h)}\}_{h\in\mathbb{N}}$ is a minimizing sequence we conclude that the limit u_* is a minimum point of $\mathcal{A}$.

There are infinitely many cones $\mathcal{K}$ and, therefore, infinitely many minimizers $u_* \in \overline{\mathcal{K}}$ of $\mathcal{A}|_{\mathcal{K}}$. However, we are interested only in classical solutions, i.e. in collision-free minimizers. Next section is devoted to the exclusion of collisions for minimizers in some particular classes of cones $\mathcal{K}$.

3.3 Collisions

Let $\mathfrak{S}$ be the set of loops with collisions. Since $\partial\mathcal{K} \subset \mathfrak{S}$, a minimizer u_* has a collision at $t = t_c$ if and only if $u_{*,1}(t_c) \in \Gamma$.
Due to the topological constraints, the proof that a minimizer u_* is collision-free cannot use Marchal's idea of averaging the action on a sphere [12], nor of averaging over a circle as in [7]. We shall exclude total collisions by level estimates, and partial collisions by local perturbations and by a uniqueness result for solutions of a class of differential equations with singular data.

3.3.1 Total collisions

For some cone $\mathcal{K}$ there exist $R \in \mathcal{R}$ and $M > 0$ such that the additional symmetry

$$u_1(t + T/M) = Ru_1(t) \tag{6}$$

is compatible with membership to $\mathcal{K}$. We can further restrict the minimization to the loops in $\mathcal{K}$ that fulfill (6). In this case, if the minimizer u_* has a total collision, then it has M total collisions per period.
We can estimate the action of u_* with the action of a homographic collision-ejection motion, with a minimal central configuration (see Propositions 5.1, 5.4 in [8]). In this way, if u_* has M collisions per period, we obtain the *a priori* estimate

$$\mathcal{A}(u_*) \geq a_{\mathcal{R},M} := 3\Big(\frac{N}{2}\Big)^{1/3}(\pi M U_0)^{2/3}T^{1/3},$$

where

$$U_0 = \min_{\rho(u)=1} U(u), \qquad \rho(u) = \frac{1}{\sqrt{N}}\Big(\sum_{h=1}^{N}|u_h|^2\Big)^{1/2}.$$

Rounded down values of $a_{\mathcal{R},M}$ for $T = 1$ and for different choices of $\mathcal{R}$, M are given in Table 1 .

Table 1 Lower bounds $a_{\mathcal{R},M}$ for loops with M total collisions ($T = 1$).

$\mathcal{R} \setminus M$	1	2	3	4	5
$\mathcal{T}$	132.695	210.640	276.017	/	/
$\mathcal{O}$	457.184	725.734	950.981	1152.032	/
$\mathcal{I}$	2296.892	3646.089	4777.728	/	6716.154

For some sequences ν, the action of the related linear piecewise loop v is lower than $a_{\mathcal{R},M}$. Therefore, minimizing the action $\mathcal{A}$ over the cones $\mathcal{K}$ defined by those sequences ν, with the additional symmetry (6), yields minimizers u_* without total collisions.

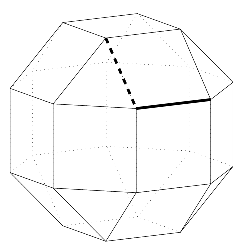


Fig. 3 For $\mathcal{R} = \mathcal{O}$ two sides of different kind are enhanced on the Archimedean polyhedron $\mathcal{Q}_{\mathcal{O}}$. The dashed side joins a triangle and a square; the solid side joins two squares.

The integrals appearing in this computation can be expressed by means of elementary functions. Indeed we have to compute a sum of integrals of (at most) two kinds due to the structure of the potential U and to the symmetry of the Archimedean polyhedron $\mathcal{Q}_{\mathcal{R}}$, which has at most two sides of different kind[1], see Figure 3. We obtain the following formula (see [8], Proposition 5.5) for the action of the loop v:

$$\mathcal{A}(v) = \frac{3}{2\cdot 4^{1/3}} N\ell^{2/3}(n_1 v_1 + n_2 v_2)^{2/3}T^{1/3}, \qquad (7)$$

[1] Indeed for $\mathcal{R} = \mathcal{T}$ all sides are of the same kind.

where $n_i, i = 1, 2$ are the numbers of sides of the two different kinds in the trajectory of v, ℓ is the length of the sides (assuming the polyhedra are inscribed in the unit sphere $\mathbb{S}^2$) and $\upsilon_i, i = 1, 2$ are the values of explicitly computable integrals, see Table 2.

Table 2 Numerical values of ℓ, υ_1, υ_2.

$\mathcal{R}$	$\mathcal{T}$	$\mathcal{O}$	$\mathcal{I}$
ℓ	1.0	0.7149	0.4479
υ_1	9.5084	20.3225	53.9904
υ_2	9.5084	19.7400	52.5762

3.3.2 Excluding partial collisions

Assume u_* has a partial collision at time $t = t_c$. Because of the symmetry, the generating particle must collide on a rotation axis, say r. Actually all the particles collide in separate clusters, and the number of particles in each colliding cluster is the order $\mathfrak{o}_\mathcal{C}$ of the maximal cyclic subgroup $\mathcal{C}$ of rotations around r in $\mathcal{R}$. We can also assume that collisions are isolated in time [2], [7].
The motion of the generating particle $t \to u_{*,1}(t)$, colliding on r at $t = t_c$, satisfies an equation of the form

$$\ddot{w} = \alpha \frac{(R_\pi - I)w}{|(R_\pi - I)w|^3} + V_1(w)\,, \qquad \alpha = \sum_{j=1}^{\mathfrak{o}_\mathcal{C}-1} \frac{1}{\sin\left(\frac{j\pi}{\mathfrak{o}_\mathcal{C}}\right)}, \tag{8}$$

where R_π is the rotation of π around the axis r and $V_1(w)$ is a smooth function.
Assume t_c=0 and let $w : (0, \bar{t}) \to \mathbb{R}^3$ be a maximal solution of (8) with $\lim_{t\to 0^+} w(t) = 0$. Then there exists $b \in \mathbb{R}$ and a unit vector n^+, with $\mathsf{n}^+ \perp r$, such that

$$\lim_{t\to 0^+} \frac{\dot{w}(t) + R_\pi \dot{w}(t)}{2} = b\mathsf{e}_r, \qquad \lim_{t\to 0^+} \frac{w(t)}{|w(t)|} = \mathsf{n}^+ \;. \tag{9}$$

The vector n^+ corresponds to the ejection limit direction. This singularity can be regarded asymptotically as a parabolic binary collision. In fact we can perform a blow-up [7], [19] by considering the rescaled functions

$$w^\lambda : [0, 1] \to \mathbb{R}^3, \qquad w^\lambda(\tau) = \lambda^{2/3} w(\tau/\lambda),$$

with $\lambda > 1/\bar{t}$. The family of functions $\{w^\lambda\}_\lambda$ converges uniformly in $[0, 1]$, as $\lambda \to +\infty$, to the parabolic ejection motion

$$s^{\alpha}(\tau)\mathsf{n}^{+}, \qquad s^{\alpha}(\tau) = (3^{3/2}/2)\alpha^{1/3}\tau^{1/3}.$$

Analogous statements hold assuming $\lim_{t\to 0^-} w(t) = 0$, with a unit vector n^-, $\mathsf{n}^- \perp r$ corresponding to the collision limit direction.

We say that a cone $\mathcal{K}$ is *simple* if the corresponding sequence $\mathfrak{t}$ does not contain a string $\tau_k \ldots \tau_{k+2\mathfrak{o}}$ such that

$$\bigcap_{j=0}^{2\mathfrak{o}} \overline{\tau_{k+j}} = p,$$

where $p \in \mathbb{S}^2$ is the pole of some rotation $R \in \mathcal{R} \setminus \{I\}$ and $\mathfrak{o}$ is the order of p.

Given a minimizing sequence $\{u^{(h)}\}_{h\in\mathbb{N}}$ we can assume that each loop $u_1^{(h)}$, describing the motion of the generating particle, when projected on $\mathbb{S}^2$, crosses the minimum number of triangles τ_k in a period (see [8], Proposition 4.4).

If the minimizer u_*, which is the limit of such a sequence, is such that $u_{*,1}$ has a collision on the axis r at time $t = t_c$, then we can define a *collision angle* θ that represents the angle between the incoming and outcoming asymptotic directions n^+, n^-, taking into account the complexity of trajectories in the minimizing sequence (this allows for example to distinguish between cases with $\theta = 0$ and $\theta = 2\pi$). Let $\mathfrak{o}_r$ be the order of the maximal cyclic group related to the collision axis r. If $\mathcal{K}$ is simple we have

$$-\frac{\pi}{\mathfrak{o}_r} \le \theta \le 2\pi.$$

If $\theta \neq 2\pi$ we can exclude partial collisions by local perturbations, that are constructed with direct or indirect arcs, and with the blow-up technique (see [8], Proposition 5.8). If $\theta = 2\pi$ we have

i) $\mathsf{n}^+ = \mathsf{n}^-$,
ii) the plane $\pi_{r,\mathsf{n}}$ generated by r, $\mathsf{n} = \mathsf{n}^{\pm}$ is fixed by some reflection $\tilde{R} \in \tilde{\mathcal{R}}$.

In this case we cannot exclude the singularity by choosing between direct and indirect arc, because the indirect arc is not available. To exclude these kind of collisions we use the following uniqueness result (see [8], Proposition 5.9) for solutions of equation (8) with singular initial data.

Proposition 2. *Let* $w_i : (0, \bar{t}_i) \to \mathbb{R}^3, \bar{t}_i > 0, i = 1, 2$ *be two maximal solutions of (8) such that* $\lim_{t\to 0^+} w_i(t) = 0$*. If* h_i*,* b_i*,* n_i*,* $i = 1, 2$ *are the corresponding values of the energy, and the values of* b *and* n^+ *given by (9), then*

$$h_1 = h_2\,, b_1 = b_2\,, \mathsf{n}_1 = \mathsf{n}_2 \qquad \textit{implies} \qquad \bar{t}_1 = \bar{t}_2\,, w_1 = w_2\ .$$

As a consequence, using the symmetry of the potential U, we have

Corollary 1. *Let* $w : (0, \bar{t}) \to \mathbb{R}^3$ *be a maximal ejection solution of (8) and* $\mathsf{n}^+ = \lim_{t\to 0^+} \frac{w(t)}{|w(t)|}$. *Assume the plane* π_{r,n^+} *generated by* r, n^+ *is fixed by some reflection* $\tilde{R}$ *in* $\tilde{\mathcal{R}}$. *Then*

$$w(t) \in \pi_{r,\mathsf{n}^+}, \ \forall t \in (0, \bar{t}) \ .$$

A similar result holds for collision solutions $w : (-\bar{t}, 0) \to \mathbb{R}^3$.
We can assume that all partial collisions of the generating particle satisfy conditions i), ii) above. Indeed different kinds of collisions are excluded by local perturbations. From Proposition 2 and Corollary 1 the generating particle must move on a reflection plane, between two rotation axes. This contradicts membership to $\overline{\mathcal{K}}$, except for particular topological constraints, defined by sequences ν winding around two axes only.
We conclude that, provided the cone $\mathcal{K}$ is simple, and the related sequence ν does not wind around one axis, nor two axes only, then minimizing the action $\mathcal{A}$ in $\mathcal{K}$ yields loops u_* free of partial collisions.

3.4 Main results

For $\mathcal{R} \in \{\mathcal{T}, \mathcal{O}, \mathcal{I}\}$ we can list a number of periodic sequences ν, determined by the elements ν_k, $k = 0 \ldots \kappa_\nu$, with κ_ν the minimal period of ν, such that there exists a T-periodic solution of the N-body problem in the same free homotopy class of ν (see [8], Theorem 4.1). For example, assuming $\mathcal{R} = \mathcal{O}$, we can consider

$$\nu \equiv [5, 1, 16, 10, 3, 8, 18, 7, 20, 23, 14, 11, 5],$$

where the numbering of vertexes of $\mathcal{Q}_{\mathcal{O}}$ is given in Figure 4. This sequence corresponds to the paths in Figure 2.
Assume $T = 1$ and let $\mathcal{K}$ be the cone associated to ν. The loops in $\mathcal{K}$ do not wind around one axis only, therefore the action $\mathcal{A}$ restricted to $\mathcal{K}$ is coercive and we can find a minimizer u_* of $\mathcal{A}|_{\mathcal{K}}$.
From the structure of ν we can restrict the minimization to the loops satisfying (6), with $M = 4$. Then by Table 1 we have

$$a_{\mathcal{R},M} \approx 1152.$$

Now consider the linear piecewise loop v defined by ν, with the same velocity on each linear path. Referring to the notation of relation (7) we have $n_1 = 8$, $n_2 = 4$, so that by Table 2 we obtain

$$\mathcal{A}(v) \approx 703.2 < a_{\mathcal{R},M},$$

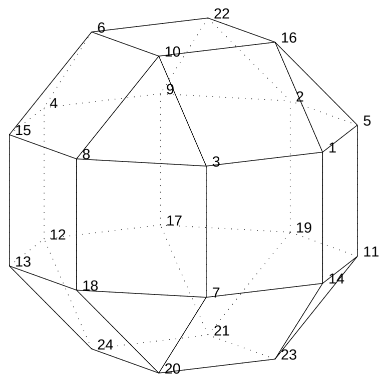


Fig. 4 The Archimedean polyhedron $\mathcal{Q}_{\mathcal{O}}$ with numbered vertexes.

so that u_* does not have total collisions.
Moreover, $\mathcal{K}$ is simple and ν does not wind around two axes only, therefore we can exclude also partial collisions.
By Palais' principle of symmetric criticality [15] we conclude that the minimizer u_* is actually a critical point of the unconstrained action functional.
Finally, using the theory of elliptic regularity we conclude that u_* is a smooth periodic solution of the N-body problem.

An interesting feature of this orbit is that the particles do not share a common center, i.e. relation (3) does not hold. This is a consequence of Theorem 4.2 in [8], implying that the projection of a collision-free minimizer $u_* \in \mathcal{K}$ on the unit sphere $\mathbb{S}^2$ crosses the minimum number of triangles compatible with membership to $\mathcal{K}$, i.e. the same triangles in the sequence $\mathfrak{t}$ that defines $\mathcal{K}$ (see Figure 2, left).

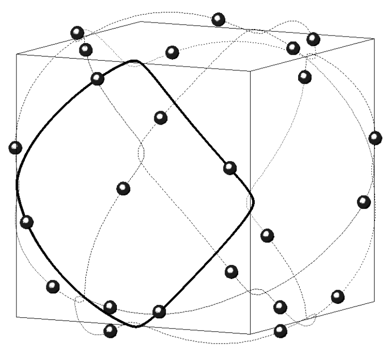


Fig. 5 Periodic motion of the $N = 24$ particles.

In Figure 5 we sketch the motion of the $N = 24$ particles. The dashed hexahedron is used as reference for the symmetry. The bodies may be divided

into 6 groups of 4 particles, each group following a choreography. The path of one of these choreographies is enhanced.
Some movies with this and similar orbits can be found at the website [9].

References

1. Barutello, V., Ferrario, D. L., Terracini, S.: On the singularities of generalized solutions to n-body-type problems, Int. Math. Res. Not., Art. ID rnn 069, 1-78 (2008)
2. Chenciner, A.: Action minimizing solutions of the Newtonian n-body problem: from homology to symmetry, ICM 2002, Beijing (2002)
3. Chenciner, A.: Symmetries and “simple” solutions of the classical N-body problem, ICMP03 (2003)
4. Chenciner, A., Montgomery, R.: A remarkable periodic solution of the three-body problem in the case of equal masses, Ann. Math. (2), **152**, 881-901 (2000)
5. Coti Zelati, V.: Periodic solutions for N-body type problems, Ann. Inst. H. Poincaré, Anal. Non Linéaire, **7/5**, 477-492 (1990)
6. Degiovanni, M., Giannoni, F., Marino, A.: Periodic solutions of dynamical systems with Newtonian type potentials, Ann. Scuola Norm. Sup. Pisa, Cl. Sci., **15**, 467-494 (1988)
7. Ferrario, D., Terracini, S.: On the existence of collisionless equivariant minimizers for the classical n-body problem, Invent. Math., **155**, 305-362 (2004)
8. Fusco, G., Gronchi, G.F., Negrini, P.: Platonic Polyhedra, Topological Constraints and Periodic Solutions of the Classical N-Body Problem, Invent. Math., **185**, 283-332 (2011)
9. `http://adams.dm.unipi.it/~gronchi/nbody.html`
10. Gordon, W.B.: A minimizing property of Keplerian orbits, American Journal of Mathematics, **99** n.5, 961-971 (1977)
11. Kepler. J.: Mysterium Cosmographicum (1621); English version: The Secret of the Universe, translated by E. J. Aiton, Abaris Books, N.Y. (1981)
12. Marchal, C.: How the method of minimization of action avoid singularities, Cel. Mech. Dyn. Ast., **83**, 325-353 (2002)
13. Poincaré, H .: Les méthodes nouvelles de la Mécanique Céleste, (1892), reprinted by Blanchard, Paris, Vol.I pag.82 (1987)
14. Moore, C.: Braids in Classical Dynamics, Phys. Rev. Lett., **70** n.24, 3675-3679 (1993)
15. Palais, R.: The principle of symmetric criticality, Comm. Math. Phys., **69** n.1, 19-30 (1979)
16. Poincaré, H.: Sur Les Solutions Périodiques et le Principe de Moindre Action, C.R.A.S, t.123, 915-918 (1896)
17. Simó, C.: New Families of Solutions in N-Body Problems, Proceedings of the third European Congress of Mathematics, Casacuberta et al. edits, *Progress in Mathematics*, **201**, 101-115 (2001)
18. Tonelli, L.: Fondamenti del calcolo delle variazioni, Zanichelli, Vol. I (1921); Vol. II (1923)
19. Venturelli, A.: Application de la minimisation de l’action au Problème de N corps dans le plan e dans l’espace, Thesis, University Paris VII (2002)
20. Wintner, A.: The Analytical Foundations of Celestial Mechanics, Princeton University press (1941)