On the possible values of the orbit distance between a near-Earth asteroid and the Earth

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Abstract

We consider all the possible trajectories of a near-Earth asteroid (NEA), corresponding to the whole set of heliocentric orbital elements with perihelion distance \( q \leq 1.3 \) au, and eccentricity \( e \leq 1 \) (NEA class). For these hypothetical trajectories we study the range of the values of the distance from the trajectory of the Earth (assumed on a circular orbit) as function of selected orbital elements of the asteroid. The results of this geometric approach are useful to explain some aspects of the orbital distribution of the known NEAs. We also show that the maximal orbit distance between an object in the NEA class and the Earth is attained by a parabolic orbit, with apsidal line orthogonal to the ecliptic plane. It turns out that the threshold value of \( q \) for the NEA class (\( q_{\text{max}} = 1.3 \) au) is very close to a critical value, below which the above result is not valid.

‘Nothing was visible, nor could be visible, to us, except Straight Lines’, E. A. Abbott, Flatland.

1 Introduction

In this paper we consider all the possible trajectories of a near-Earth asteroid (NEA), that is the whole set of trajectories with perihelion distance \( q \leq 1.3 \) au, and eccentricity \( e \leq 1 \). We study the range of the values of the orbit distance between hypothetical asteroids on these trajectories and the Earth, which is assumed on a circular orbit.

The orbit distance between an asteroid and the Earth, here denoted by \( d_{\text{min}} \), is the minimum value of the distance between two points on the two orbits; it
is also called MOID\(^1\) in the literature. This distance plays an important role to understand whether an asteroid can impact the Earth: for this purpose the category of potentially hazardous asteroids (PHAs) has been introduced (see [2]), that are NEAs with \(d_{\text{min}} \leq 0.05\) au, and absolute magnitude \(H \leq 22\). The value of \(H\) is related to the size of the asteroid and the constraint \(H \leq 22\) is set to take into account objects which are large enough to produce serious damages in case of impact with our planet.

The conversion between size and absolute magnitude of an asteroid strongly depends on the albedo (measuring the reflectivity properties of the body), whose value is badly known for almost all asteroids. For this reason we prefer to speak of bright \((H \text{ small})\) and faint asteroids \((H \text{ large})\).

The orbit distance \(d_{\text{min}}\) is also important to understand whether a faint asteroid can be observed from the Earth: hereafter we deal with this issue. If \(d_{\text{min}}\) is large, then the asteroid is hard to detect. On the contrary, if \(d_{\text{min}}\) is small, in most cases the asteroid will get close enough to the Earth and will be detected, provided we wait long enough.

Of course other factors intervene in an asteroid detection: weather, solar elongation of the observed body, etc.; however, we shall show that a purely geometric argument is enough to explain some selection effects in the orbital distribution of the observed population of near-Earth asteroids.

In Figure 1 we plot the values of the perihelion distance and the perihelion argument, that is the pairs \((q, \omega)\), for all the 9220 known NEAs to the date of October 21, 2012. The gray dots represent asteroids with \(H \leq 22\), while the black dots are those with \(H > 22\). We immediately observe that most of the fainter asteroids are grouped together in a peculiar way.

In Figure 2 we plot the values of the pairs \((q, d_{\text{min}})\): also here the gray dots are asteroids with \(H \leq 22\), the black ones with \(H > 22\). In this case a V-shaped structure appears, composed of two lines: the fainter NEAs accumulate towards the line with \(q > 1\) au, while that with \(q < 1\) au is mostly due to the brighter NEAs.

In this paper we shall prove some facts concerning the distance between two confocal conics, and we shall see that these purely geometric results can give an explanation of the features appearing in Figures 1, 2.

The structure of the paper is the following. In Section 2 we introduce preliminary facts and notations. In Section 3 we prove the geometric results. Section 4 is devoted to use these results and explain the orbital distribution of the known NEAs. In Section 5 we discuss the perihelion distance threshold of the NEA class, from the point of view of the orbit distance. Finally, in Appendix A we put some complementary results and computations.

\(^1\)Minimum Orbit Intersection Distance
Figure 1: Orbital distribution of the known NEAs in the plane $(q, \omega)$. The black dots correspond to the fainter asteroids, that have absolute magnitude $H > 22$. The gray dots represent all the others.

Figure 2: Orbital distribution of the known NEAs in the plane $(q, d_{\text{min}})$. The black dots correspond to the fainter asteroids ($H > 22$).
2 Definitions

2.1 Preliminaries

We denote by $\mathcal{N}$ the set of possible trajectories of NEAs: these can be described by cometary orbital elements $E = (q, e, I, \Omega, \omega)$ with the constraints

$$0 \leq q \leq q_{\text{max}} = 1.3 \text{ au}, \quad 0 \leq e \leq 1.$$  

(1)

Here $q$ is the perihelion distance, $e$ the eccentricity, $I$ the inclination, $\Omega$ the longitude of the ascending node and $\omega$ the argument of perihelion. We adopt these set of elements because they naturally lend themselves to the description of parabolic orbits, which are included in the set $\mathcal{N}$.

On the other hand, if $q = 0$ the trajectory is either pointwise or rectilinear, and using cometary orbital elements we can neither distinguish these two cases, nor characterize two rectilinear trajectories with different length. Moreover, if $I = 0, \pi$ or $e = 0$ the angle $\omega$ is not defined. If $I = 0$ also $\Omega$ is not defined, but this orbital element will not play a role in the following.

We assume that the Earth moves on a circular orbit, whose elements $E' = (q', e', I', \Omega', \omega')$ are set as follows:

$$q' = 1 \text{ au}, \quad e' = I' = \Omega' = \omega' = 0.$$  

Hereafter we will denote by $\mathcal{A}$ the trajectory of an asteroid in the NEA class, and by $\mathcal{A}'$ that of the Earth.

Moreover, for $q \neq 0$ and $I \neq 0, \pi$, we introduce the ascending/descending nodal distances

$$d_{\text{nod}}^\pm = q - \frac{q(1 + e)}{1 \pm e \cos \omega}.$$  

(2)

2.2 The orbit distance $d_{\text{min}}$

Let us consider two celestial bodies on confocal Keplerian orbits. We fix a reference frame, with origin in the common focus and let $(E, v), (E', v')$ be the sets of orbital elements of the bodies. Here $E, E'$ describe the trajectories of the orbits and $v, v'$ are parameters along them, e.g. the true anomalies $f, f'$. We denote by $\mathcal{E} = (E, E')$ the two-orbit configuration, and by $V = (v, v')$ the vector of the orbit parameters. Moreover, we write $\mathcal{X} = \mathcal{X}(E, v), \mathcal{X}' = \mathcal{X}'(E', v')$ for the Cartesian coordinates of the two bodies.

For a given two-orbit configuration $\mathcal{E}$, we introduce the Keplerian distance function $d$, defined by

$$V \mapsto d(\mathcal{E}, V) = |\mathcal{X} - \mathcal{X}'|,$$

(3)

where $V \in T^2$ (the two dimensional torus) if $e < 1$, $V \in (-\pi, \pi) \times S^1$ if $e = 1$. Here $| \cdot |$ is the Euclidean norm in $\mathbb{R}^3$.

The local minimum points of $d$ can be found by computing all the stationary points of $d^2$, as in [6], or [1], where the authors use algebraic tools such as resultants and Gröbner’s bases. See also [5], [11].
Apart from the case of two concentric coplanar circles, or two overlapping ellipses, the function \( d^2 \) has finitely many stationary points [6]. We can find configurations with up to 4 local minima of \( d^2 \): this is thought to be the maximum possible.

Let \( V_h = V_h(\mathcal{E}) \) be a local minimum point of \( V \mapsto d^2(\mathcal{E}, V) \). Then, following [7], we consider the maps

\[
\begin{align*}
\mathcal{E} & \mapsto d_h(\mathcal{E}) = d(\mathcal{E}, V_h) \quad \text{(local minimal distance)}, \\
\mathcal{E} & \mapsto d_{\min}(\mathcal{E}) = \min_h d_h(\mathcal{E}) \quad \text{(orbit distance)}. 
\end{align*}
\]  

(4)

For each choice of the configuration \( \mathcal{E} \), \( d_{\min}(\mathcal{E}) \) gives the orbit distance.

The maps \( d_h, d_{\min} \) are not differentiable where they vanish. This singularity has been studied in [7] to define a meaningful uncertainty of \( d_h, d_{\min} \), with possibly negative values of these maps. However, this will not constitute a problem in the present work.

The maps \( d_h \) may have other singularities due to bifurcations. Moreover, \( d_{\min} \) can lose regularity when two local minima exchange their role as absolute minimum. A detailed analysis of these phenomena is still lacking. The occurrence of bifurcations will not be a problem as well, since at bifurcation points the derivative of \( d_h \) with respect to an orbital element can be extended with regularity, see Figure 3 (b). On the contrary, possible exchanges of local minimum points as absolute minimum make things more difficult because the monotonicity properties of the maps \( d_h \) may not hold for \( d_{\min} \), see Figure 3 (a).

We organize the proofs in the paper so that there is no need to know if and where such a singularity occurs.

### 3 Geometric results

The maximum of the orbit distance \( d_{\min} \) in the NEA class \( \mathcal{N} \) does exist. Below we identify the NEA orbits attaining this maximum and completely characterize the possible values of \( d_{\min} \) assuming the values of some orbital elements.
Since $\mathcal{A}'$ is circular we can set $\Omega = 0$ in computing the orbit distance between $\mathcal{A}$ and $\mathcal{A}'$. Thus the elements of $\mathcal{E}$ that can vary are only $q, e, I, \omega$. Moreover, taking advantage of the symmetry, we can restrict our analysis to $\omega \in [0, \pi/2]$, $I \in [0, \pi/2]$.

### 3.1 The maximal orbit distance

If $\mathcal{A}$ is either pointwise or rectilinear, then the maximal orbit distance is $q' = 1$ au. Assume $q \neq 0$. We parametrize the trajectories $\mathcal{A}, \mathcal{A}'$ as follows:

$$x = r \cos(f + \omega), \quad y = r \sin(f + \omega) \cos I, \quad z = r \sin(f + \omega) \sin I,$$

with

$$r = \frac{q(1 + e)}{1 + e \cos f},$$

and

$$x' = q' \cos f', \quad y' = q' \sin f', \quad z' = 0.$$  

This choice of coordinates implies that the mutual nodal line coincides with the $x$ axis. We denote the position vector of the asteroid by

$$\mathbf{r} = \mathbf{r},$$

where

$$\mathbf{r} = (\cos(f + \omega), \sin(f + \omega) \cos I, \sin(f + \omega) \sin I).$$

The squared distance function is

$$d^2(\mathcal{E}, V) = (x - x')^2 + (y - y')^2 + z^2 = \frac{q^2(1 + e)^2}{(1 + e \cos f)^2} + q'^2 - \frac{2qq'(1 + e)}{1 + e \cos f} [\cos f' \cos(f + \omega) + \sin f' \sin(f + \omega) \cos I].$$

The orbit distance $d_{\text{min}}$ is the value of $d$ at one of the solutions of

$$\frac{\partial d^2}{\partial f} = 0, \quad \frac{\partial d^2}{\partial f'} = 0,$$

with

$$\frac{\partial d^2}{\partial f} = 2(x - x') \frac{\partial x}{\partial f} + 2(y - y') \frac{\partial y}{\partial f} + 2z \frac{\partial z}{\partial f}$$

and

$$\frac{\partial d^2}{\partial f'} = -2(x - x') \frac{\partial x'}{\partial f'} = 2(y - y') \frac{\partial y'}{\partial f'} = 2(xy' - x'y),$$

With this parametrization we obtain only pointwise trajectories for $q = 0$. If $e = 0$ or $I = 0$ every choice of $\omega$ produces the same trajectory.

Since $\frac{\sin f}{\sin I} = \frac{\sin(f + \omega)}{1 + e \cos f}$, we can consider the expression $\frac{\sin f}{\sin I}$ meaningful also for $I = 0$.  

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6
where we have used the relations
\[
\begin{align*}
\frac{\partial x}{\partial f} &= -\frac{z}{\sin I} + xg(f, e), \\
\frac{\partial y}{\partial f} &= x\cos I + yg(f, e), \\
\frac{\partial z}{\partial f} &= x\sin I + zg(f, e),
\end{align*}
\]
with
\[
g(f, e) = \frac{e\sin f}{1 + e\cos f}, \tag{6}
\]
and
\[
\frac{\partial x'}{\partial f'} = -y', \quad \frac{\partial y'}{\partial f'} = x'.
\]
Now we prove the following result:

**Proposition 1.** We have
\[
\max_{N} d_{\min} = D(\bar{x}),
\]
where
\[
D(x) = \sqrt{(x - q')^2 + \left(\frac{x^2 - 4q_{\max}^2}{4q_{\max}}\right)^2} \tag{7}
\]
and \(\bar{x}\) is the (unique) real solution of
\[
x^3 + 4q_{\max}^2x - 8q'q_{\max}^2 = 0. \tag{8}
\]
We have \(\bar{x} \approx 1.5\) au and \(D(\bar{x}) \approx 1.0011\) au.

**Proof.** We easily find that if \(A\) is a pointwise or a rectilinear trajectory then the value of the orbit distance is at most \(q' = 1\) au.

Now we consider trajectories with \(q \neq 0\) (neither rectilinear nor pointwise). Denote by \(A_\ast\) a trajectory in \(N\) attaining the maximal orbit distance. We divide the proof into 5 steps.

**Step 1.** We can choose \(I = \pi/2\).

We compute the derivative of a squared local minimal distance:
\[
\frac{\partial d_h^2}{\partial I}(\mathcal{E}) = \frac{\partial d^2}{\partial I}(\mathcal{E}, V_h) + \frac{\partial d^2}{\partial V}(\mathcal{E}, V_h) \frac{\partial V_h}{\partial I}(\mathcal{E}). \tag{9}
\]
Since \(V_h = (f_h, f'_h)\) is a stationary point of \(V \mapsto d^2(\mathcal{E}, V)\), (9) reduces to
\[
\frac{\partial d_h^2}{\partial I}(\mathcal{E}) = \frac{\partial d^2}{\partial I}(\mathcal{E}, V_h) = 2y'(f'_h)z(f_h), \tag{10}
\]

which has the same sign as \( \sin f_h + \omega \). Therefore, if \( d_h = d_{\text{min}} \), then \( d_h \) is a non-decreasing function of \( I \) for \( 0 < I < \pi/2 \); in fact for \( 0 < I < \pi/2 \) the minimal distance can be attained only if \( y'(f'_{\text{min}})z(f_{\text{min}}) \geq 0 \) because we have

\[
d_h^2 = q^2 + r^2(f_h) - 2q'r(f_h)[\cos(f_h + \omega) \cos f'_h + \sin(f_h + \omega) \sin f'_h \cos I]
\]

and, if \( \sin(f_h + \omega) \sin f'_h < 0 \), then by changing \( f'_h \) with \( -f'_h \) we would obtain a value of \( d \) smaller than \( d_h \). This implies that the maximal value of \( d_h \) is attained for \( I = \pi/2 \). Note that, even if there were an exchange of local minimum points as absolute minimum when \( I \) varies, then from the computations above this would not matter: in fact all the local minimal distances have the same monotonicity properties as functions of \( I \).

**Step 2.** We must have \( y'(f'_{\text{min}}) = 0 \). Moreover, if \( e \neq 0 \), then \( A_* \) must have either \( \omega = 0 \) or \( \omega = \pi/2 \).

First we observe that the second equation in (5) for \( I = \pi/2 \) gives \( x(f_h)y'(f'_h) = 0 \). We exclude the solutions with \( x(f_h) = 0 \), that corresponds to two points \( P \pm \) on the \( z \) axis, since they cannot attain the minimal distance \( d_{\text{min}} \). In fact, we observe that if \( x(f_h) = 0 \) then \( d_h = \sqrt{z_h^2 + q'_h^2} \) with \( z_h = z(f_h) \geq q \), whatever the value of \( f'_h \). On the other hand, since \( q \neq 0 \),

\[
d_{\text{min}} \leq \min |d^\pm_{\text{nod}}| < \min \sqrt{q^2 + q^2 \left( \frac{1 + e}{1 \pm e \cos \omega} \right)^2} \leq \sqrt{q^2 + q^2} < \sqrt{q^2 + z_h^2} = d_h .
\]

Thus we must have \( y'(f'_{\text{min}}) = 0 \) and, if \( e \neq 0 \), we have to choose \( \omega \) that maximizes the minimum of the two distances between the nodal points \( Q_{\pm} \equiv (\pm q', 0, 0) \) and \( A \). We select one nodal point, say \( Q_+ \); by symmetry, the discussion for the distance between \( A \) and \( Q_- \) can be included by extending to \( \omega \in [0, \pi] \) the study of the distance between \( A \) and \( Q_+ \).

This distance is given by minimizing the function

\[
f \mapsto d_{Q_+}^2(f, e, \omega) = r^2 - 2rq' \cos(f + \omega) + q'^2,
\]

with

\[
r = r(f, e) = \frac{q(1 + e)}{1 + e \cos f}
\]

and \( f \in S^1 \) if \( e < 1 \), \( f \in (-\pi, \pi) \) if \( e = 1 \). The stationarity condition for \( f \mapsto d_{Q_+}^2(f, e, \omega) \) gives

\[
\frac{\partial d_{Q_+}^2}{\partial f} = 2 \left( \frac{\partial r}{\partial f} - \frac{\partial r}{\partial f} q' \cos(f + \omega) + rq' \sin(f + \omega) \right) = 0 , \quad (11)
\]

with

\[
\frac{\partial r}{\partial f} = rg , \quad (12)
\]
and $g$ defined as in (6). Let $f_h$ be a minimum point of $f \mapsto d^2_{Q_+}(f, e, \omega)$; then we write
\[
d^2_{Q_+, h}(e, \omega) = d^2_{Q_+}(f_h(e, \omega), e, \omega),
\]
that gives the local minimal distance from $Q_+$ assuming, as we did, $I = \pi/2$.

The derivative
\[
\frac{\partial d^2_{Q_+, h}}{\partial \omega}(e, \omega) = \frac{\partial d^2_{Q_+}}{\partial \omega}(f_h(e, \omega), e, \omega) = 2r_h q' \sin(f_h + \omega),
\]
with $r_h = r(f_h, e)$, can vanish only if $\sin(\omega + f_h) = 0$. By using (11) and (12) we obtain that
\[
\frac{\partial d^2_{Q_+, h}}{\partial \omega}(e, \omega) = 0 \text{ if and only if } g_h(r_h \mp q') = 0,
\]
with $g_h = g(f_h, e)$. Thus the derivative (13) can vanish only for $g(f_h, e) = 0$, that together $\sin(f_h + \omega) = 0$ gives $\sin \omega = 0$ (since $e \neq 0$), or for $r(f_h, e) = q'$, that implies $d_{Q_+, h} = 0$.

By a classical result on the distance between a point and a conic section in the plane (see Section A.1 for details), for each $\omega \in [0, \pi]$ the only exchange of local minimum points $f_h(e, \omega)$ as absolute minimum of $d_{Q_+}$ can occur for $\omega = 0, \pi$.

Let us fix a value for $e \neq 0$. From the discussion above follows that there are 3 possible cases (see Figure 4): 1) $\omega \mapsto \min f d_{Q_+}(e, \omega)$ is increasing over $(0, \pi)$; 2) $\omega \mapsto \min f d_{Q_+}(e, \omega)$ is decreasing over $(0, \pi)$; 3) there exists $\bar{\omega} \in (0, \pi)$ such that $\omega \mapsto \min f d_{Q_+}(e, \omega)$ is decreasing over $(0, \bar{\omega})$, vanishes at $\omega = \bar{\omega}$ and is increasing over $(\bar{\omega}, \pi)$. 

Figure 4: Some possible graphs of the distances between $Q_\pm$ and $A$, denoted by $\min f d_{Q_\pm}$, as function of $\omega$. The plot of $\min f d_{Q_-}$ is dashed.
Moreover, by symmetry we have
\[ \min_f d_{Q^-}(e, \omega) = \min_f d_{Q^+}(e, \pi - \omega) . \]

In Figure 4 we show some of the possible graphs of \( \min_f d_{Q^+} \) and \( \min_f d_{Q^-} \).

We recall that
\[ d_{min} = \min \{ \min_f d_{Q^+}, \min_f d_{Q^-} \} , \]

therefore we conclude that, for \( e \neq 0 \), the map
\[ [0, \pi/2] \ni \omega \mapsto d_{min}(e, \omega) \]
attains its maximum value either for \( \omega = 0 \) or for \( \omega = \pi/2 \).

**Step 3.** \( \mathcal{A}_* \) must have all the nodes external to \( \mathcal{A}' \), hence \( \omega = \pi/2 \).

If not, we have \( d_{min} \leq 1 \) au and this cannot be a configuration which maximizes \( d_{min} \) in \( \mathbb{N} \). In fact the parabolic trajectory \( \mathcal{P} \), with \( I = \pi/2, \omega = \pi/2, q = q_{max} \), gives the larger value \( d_{min} = D(\bar{x}) \approx 1.0011 \) au, defined by (7), (8): the equation of the trajectory of \( \mathcal{P} \) in the plane \((x, z)\) can be written as
\[ z = \frac{x^2 - 4q_{max}^2}{4q_{max}} , \]
thus the distance of the points of the parabola from \((x, z) \equiv (q', 0)\) is given by (7). The stationary points equation for the function \( D(x) \) defined in (7) is (8), and it has only one solution \( \bar{x} > 0 \), which corresponds necessarily to the absolute minimum point.

Since the nodes of \( \mathcal{A}_* \) are external to \( \mathcal{A}' \) we exclude the value \( \omega = 0 \), that gives \( d_{min} = q - q' \), which is smaller than the orbit distance for \( \omega = \pi/2 \).

**Step 4.** \( \mathcal{A}_* \) must have \( q = q_{max} \).

By step 2 we can consider in the plane \((x, z)\) the problem of minimizing the distance between a point in \( \mathcal{A} \) and a nodal point \( Q_{\pm} \). If we change \( q \) into \( q + \delta q \), with \( \delta q > 0 \), the modified trajectory encloses the original one. By step 3 the nodes of \( \mathcal{A}_* \) are external to \( \mathcal{A}' \), hence every line joining either \( Q_+ \) or \( Q_- \) to a point of the modified trajectory must cross the original trajectory, and we conclude that both distances of \( Q_{\pm} \) from \( \mathcal{A} \) increase with \( q \). This implies that the maximal value of every \( d_{h} \), hence also of \( d_{min} \), is attained for \( q = q_{max} \).

**Step 5.** \( \mathcal{A}_* \) must have \( e = 1 \).

We have
\[ \frac{\partial r}{\partial e} = \frac{\partial r}{\partial e} \bar{r} , \quad \frac{\partial r}{\partial e} = \frac{q(1 - \cos f)}{(1 + e \cos f)^2} \geq 0 . \] (14)

Moreover, by step 3 the nodes of \( \mathcal{A}_* \) are external to \( \mathcal{A}' \). Using \( I = \pi/2 \) (which holds by step 1) and \( y'(f_{min}') = 0 \) (by step 2) we conclude that the maximum value of \( d_{min} \) is attained at \( e = 1 \).
Figure 5: Graph of the function \((q, \omega) \mapsto \max_{\mathcal{D}_1} d_{\min}(q, \omega)\).

Figure 6: Level curves of \((q, \omega) \mapsto \max_{\mathcal{D}_1} d_{\min}(q, \omega)\). We also plot the curve \(\gamma\) (enhanced in the figure) defined in (18).
3.2 Optimal bounds for \( d_{\min} \)

We study the range of the possible values of \( d_{\min} \) by assuming the values of selected orbital elements.

First we establish the possible values of \( d_{\min} \) as function of \((q, \omega)\).

**Proposition 2.** Let \( D_1 = \{(e, I) : 0 \leq e \leq 1, 0 \leq I \leq \frac{\pi}{2}\} \), \( D_2 = \{(q, \omega) : 0 < q \leq q_{\max}, 0 \leq \omega \leq \frac{\pi}{2}\} \). For each choice of \((q, \omega) \in D_2\) we have

\[
\begin{cases}
\min_{(e, I) \in D_1} d_{\min} = \max\{0, q - q'\} \\
\max_{(e, I) \in D_1} d_{\min} = \max\{q' - q, \delta_\omega(q, \omega)\}
\end{cases}
\tag{15}
\]

where \( \delta_\omega(q, \omega) \) is the distance between \( A' \) and \( A \) with \( e = 1, I = \pi/2 \):

\[
\delta_\omega(q, \omega) = \sqrt{(\xi - q' \sin \omega)^2 + \left(\frac{\xi^2 - 4q^2}{4q} + q' \cos \omega\right)^2}, \tag{16}
\]

with \( \xi = \xi(q, \omega) \) the unique real solution of

\[
x^3 + 4q(q + \cos \omega)x - 8q'q^2 \sin \omega = 0.
\]

**Proof.** 

*Lower bound.* By step 1 in Proposition 1 we can choose \( I = 0 \). If \( q \leq q' \), for suitable choices of \( e \) we have \( d_{\min} = 0 \). If \( q > q' \) then \( d_{\min} \) is always greater than or equal to \( q - q' \) and it is equal to \( q - q' \) for \( e = 0 \).

*Upper bound.* By step 1 in Proposition 1, given \((q, \omega)\) we can choose \( I = \pi/2 \). Thus we have to properly choose only the value of \( e \). By step 2 in Proposition 1 we have \( y'(f_{\min}') = 0 \). Moreover the relation

\[
\frac{\partial}{\partial e} d_{\nod}^\pm = -\frac{q(1 \pm \cos \omega)}{(1 \pm e \cos \omega)^2} \leq 0
\]

holds so that, given \((q, \omega)\), if there exists \( \hat{e} \in [0, 1) \) such that the nodes of \( A \) with \( e = \hat{e} \) are external to \( A' \), then the nodes remain external for each \( A \) with \( e \in [\hat{e}, 1] \). Therefore, for each choice of \((q, \omega)\), the maximal value of \( d_{\min} \), for \( e \) such that the nodes of \( A \) are external to \( A' \), is attained for \( e = 1 \). This can be proven as in step 5 of Proposition 1. On the other hand, the maximal value of \( d_{\min} \), for \( e \) such that at least one node of \( A \) is internal to \( A' \), is attained for \( e = 0 \). In fact, in case of an internal node we necessarily have \( q < q' \), so that the maximal value of \( d_{\min} \) is \( q' - q \). In fact we have

\[
d_{\min} \leq \min |d_{\nod}^\pm| \leq q' - q
\]

and \( d_{\min} = q' - q \) for \( e = 0 \). \( \square \)
Remark 1. We note that, if the nodes of $\mathcal{A}$ are external to $\mathcal{A}'$, then necessarily $q > q'/2$. In fact we can compute

$$\min\{q \in [0, q_{\max}] : \exists (e, \omega) \in [0, 1] \times [0, 2\pi] : \max d_{\text{nod}}^e \leq 0\}$$

from the relations

$$0 \geq \min_{e \in [0, 1]} \min_{\omega \in [0, \pi]} \max \left(q' - \frac{q(1 + e)}{1 + e \cos \omega}\right) =$$

$$= \min_{e \in [0, 1]} \min_{\omega \in [0, \pi/2]} \left(q' - \frac{q(1 + e)}{1 + e \cos \omega}\right) =$$

$$= \min_{e \in [0, 1]} (q' - q(1 + e)) = q' - 2q.$$

In Figure 5 we plot the graph of the function $(q, \omega) \mapsto \max_{\text{D1}} d_{\text{min}}(q, \omega)$, and in Figure 6 we show some level curves of $\max_{\text{D1}} d_{\text{min}}$. In the latter we also draw the curve $\gamma$ which separates the region, in the plane $(q, \omega)$, where the orbits maximizing $d_{\text{min}}$ have $e = 0$, from the region where such orbits have $e = 1$. To explicitly compute $\gamma$, consider the system

$$\begin{cases}
(q' - q)^2 = (x - q' \sin \omega)^2 + \left(\frac{x^2 - 4y}{4q} + q' \cos \omega\right)^2 \\
x^3 + 4q(x + \cos \omega)x - 8q^2q' \sin \omega = 0
\end{cases} \quad (17)$$

We use resultant theory, see [4], to eliminate the variable $x$, and obtain the algebraic curve

$$2q^4 + 2q'(-5 + 7y)q^3 - 2q^2(3y + 22)(y - 1)q^2 +$$

$$+ q^3(y^2 + 13y^2 + 9y - 27)q - 2q^4y^3 = 0 \quad (18)$$

with $y = \cos \omega$. Additional details on this computation are given in Section A.2.

Now we introduce a partition of $\{(q, e) \in (0, q_{\max}] \times [0, 1]\}$ in 3 regions, an inner region (IR), a crossing region (CR) and an outer region (OR):

$$\begin{align*}
\text{IR} & = \{(q, e) : Q < q'\}, \\
\text{CR} & = \{(q, e) : Q \geq q', q \leq q'\}, \\
\text{OR} & = \{(q, e) : q > q'\},
\end{align*}$$

where $Q = q(1 + e)/(1 - e)$ is the aphelion distance (see Figure 8). In regions IR, OR the trajectory $\mathcal{A}$ is totally inside, respectively outside, the ball $B(0, q')$ with centre $O$ and radius $q'$. In region CR, crossings of $\mathcal{A}$, $\mathcal{A}'$ are possible. Region CR can be divided into 2 sub-regions:

$$\begin{align*}
\text{CR}_1 & = \{(q, e) : Q \geq q', q \leq q', q(1 + e) \leq q'\}, \\
\text{CR}_2 & = \{(q, e) : Q \geq q', q \leq q', q(1 + e) > q'\}.
\end{align*}$$

According to the value of $I, \omega$, with $I \neq 0$, in region CR$_1$ we have either two internal nodes, or only one internal; in region CR$_2$ we have either two external nodes, or only one internal.

Now we describe the possible values of $d_{\text{min}}$ as function of $(q, e)$.  

\[\text{Page 13}\]
Figure 7: Graph of the function \((q, e) \mapsto \min_{D_3} d_{\text{min}}(q, e)\).

Figure 8: Partition of \(\{(q, e) \in (0, q_{\text{max}}] \times [0, 1]\) in outer region (OR), crossing region (CR, composed by CR\(_1\) and CR\(_2\)) and inner region (IR). We also plot the level curves of \((q, e) \mapsto \min_{D_3} d_{\text{min}}(q, e)\).
Figure 9: Graph of the function \((q, e) \mapsto \max_{D} d_{\text{min}}(q, e)\).

Figure 10: Level curves of \((q, e) \mapsto \max_{D} d_{\text{min}}(q, e)\).
We observe that for \((q, e)\), \(d\) holds also for \(e\) between \(A\) where \(Q\) holds.

Let Proposition 4. In Proposition 1 we have \(y_\delta\) coincides with the second relation in (19).

\[ \delta_e(q, e) = \left[ (\xi - q')^2 + \frac{q^2e^2}{(1-e)^2} - 2\frac{qe}{(1-e)} \sqrt{\frac{q^2}{(1-e)^2} - \frac{\xi^2}{1-e^2} + \frac{q^2}{(1-e)^2} - \frac{\xi^2}{1-e^2}} \right] \]

where \(\xi = \xi(q, e)\) is the unique real positive solution of

\[ e^4x^2 + 2q^2e^2(1-e^2)x^3 + (1+e)^2(q^2(1-e^2) + q^2e^2)x^2 - 2q^2e^2(1+e)^2x - q^2q^2(1-e^2)(1+e)^2 = 0. \]

See Section A.3 for the details of this computation.

Proof. Lower bound. By step 1 in Proposition 1 we can choose \(I = 0\). Therefore, if \((q, e) \in \mathbb{R}\) we have \(d_{\min} = q' - Q\); if \((q, e) \in \mathbb{C}\) we have \(d_{\min} = 0\); if \((q, e) \in \mathbb{O}\) we have \(d_{\min} = q - q'\).

Upper bound. By step 1 in Proposition 1 we can choose \(I = \pi/2\). By step 2 in Proposition 1 we have \(y'(f_{\min}) = 0\). If \(e = 0\), for every value of \(\omega\) we get \(d_{\min} = \lfloor q - q' \rfloor\). If \(e \neq 0\), by the same step 2 we must choose either \(\omega = 0\) or \(\omega = \pi/2\). For \(\omega = 0\) we have \(d_{\min} = \min\{|Q - q'|, |q' - q|\}\), for \(\omega = \pi/2\) we have \(d_{\min} = \delta_e(q, e)\). We summarize the result with the following formula, that holds also for \(e = 0\):

\[ \max_{(I, \omega) \in D_3} d_{\min} = \max\{\min\{|Q - q'|, |q' - q|\}, \delta_e(q, e)\}. \]  \((20)\)

We observe that for \((q, e) \in \mathbb{R}\) we have \(\delta_e(q, e) \geq q' - Q\) and for \((q, e) \in \mathbb{O}\) we have \(\delta_e(q, e) \geq q - q'\). If \((q, e) \in \mathbb{C}\) then \(q \leq q'\) and \(Q \geq q'\), hence (20) actually coincides with the second relation in (19).

Now we describe the possible values of \(d_{\min}\) as function of \((q, I)\).

Proposition 4. Let \(D_5 = \{(e, \omega) : 0 \leq e \leq 1, 0 \leq \omega \leq \frac{\pi}{2}\}\), \(D_6 = \{(q, I) : 0 < q \leq q_{\max}, 0 \leq I \leq \frac{\pi}{2}\}\). For each choice of \((q, I) \in D_6\) we have

\[ \begin{align*}
\min_{(e, \omega) \in D_5} d_{\min} &= \max\{0, q - q'\} \\
\max_{(e, \omega) \in D_5} d_{\min} &= \max\{q' - q, \delta_I(q, I)\}
\end{align*} \]  \((21)\)

where \(\delta_I(q, I)\) is the distance between \(A'\) and \(A\) with the constraints \(e = 1, \omega = \pi/2\) (see Section A.4).
Figure 11: Graph of the function \((q, I) \mapsto \max_{\mathcal{D}_5} d_{\text{min}}(q, I)\).

Figure 12: Level curves of \((q, I) \mapsto \max_{\mathcal{D}_5} d_{\text{min}}(q, I)\).
Figure 13: Node crossing curves and partition of the domain $D_5$ for $q = 0.7$ au.

The asterisk $*$ (upper right corner) indicates the point attaining the maximal value of $d_{\text{min}}$ in the region with both nodes external. The maximal value of $d_{\text{min}}$ in the region with at least one internal node is attained for $e = 0$.

Proof. Lower bound. If $q > q'$ then $A$ is external to the ball $B(0, q')$ with centre $O$ and radius $q'$, so that $d_{\text{min}} \geq q - q'$. If we select $e = 0$ we obtain just $d_{\text{min}} = q - q'$.

If $q \leq q'$, for each $I$ we can find values of $(e, \omega)$ such that $d_{\text{min}} = 0$.

Upper bound. If $I = 0$ the maximal orbit distance is $|q' - q|$ and corresponds to the second relation in (21) since, for $q > q'$, $\delta_I(q, 0) = q - q'$.

Assume now $I \neq 0$. If $q < q'/2$ then there is at least a node of $A$ which is internal to $A'$: in this case the maximal value of $d_{\text{min}}$ is $q' - q$ and is attained for $e = 0$. In fact for each $(e, \omega)$ we have $d_{\text{min}} \leq \min |d_{\text{nod}}|$ and, in this case, $\min |d_{\text{nod}}| \leq q' - q$.

If $q \geq q'/2$ we consider two subsets of $D_5$: the first, denoted by $D_{5,1}$, is the set of values of $(e, \omega)$ such that at least one node of $A$ is internal to $A'$ (white and bright-shadowed regions in Figure 13, where $q = 0.7$ au). The second subset, denoted by $D_{5,2}$, is the set of values of $(e, \omega)$ such that all the nodes of $A$ are external (dark-shadowed region). In the boundary between $D_{5,1}$ and $D_{5,2}$, defined by the node crossing curves $d_{\text{nod}} = 0$, we have $d_{\text{min}} = 0$. Note that, if $q > q'$ the nodes are always external for every choice of $(e, \omega)$.

If we restrict $d_{\text{min}}$ to $D_{5,1}$ the proof works as in the case $q < q'/2$, and we obtain

$$\max_{D_{5,1}} d_{\text{min}} = q' - q,$$

which is attained for $e = 0$.

Now we prove that the maximal value of $d_{\text{min}}$ restricted to $D_{5,2}$ is given by the orbit distance of a parabolic orbit $(e = 1)$, with $\omega = \pi/2$.
In the proof we shall use the following geometric property of an absolute minimum point of $d^2$. Let $P$ be a point on $A$ attaining the minimal value $d_{\text{min}}$. Then we can consider a circular torus $T$ (see Figure 14), with parametric equations

$$
\begin{align*}
x &= (q' + d_{\text{min}} \cos \theta) \cos \phi, \\
y &= (q' + d_{\text{min}} \cos \theta) \sin \phi, \\
z &= d_{\text{min}} \sin \theta,
\end{align*}
$$

with $\theta, \phi \in S^1$. The point $P$ must lie on the torus $T$ and the remaining part of the trajectory $A$ must lie outside $T$ (because the nodes are external), being allowed to touch tangentially the torus in some other point.

Let $Q$ be the corresponding point on $A'$ attaining the minimal distance, i.e. $|P - Q| = d_{\text{min}}$. By the property of the stationary points of $d^2$, since $A'$ is circular, $P$ must lie on a circle with centre $Q$ and radius $d_{\text{min}}$, which is obtained as section of $T$, orthogonal to $A'$ (see Figure 15). Moreover, since the nodes of $A$ are external to $A'$, the $\theta$ coordinate of $P$ is subject to the constraint

$$
\cos \theta \geq 0.
$$

In fact, since in $D_{5,2}$ the nodes of $A$ are external, with $d_{\text{nod}}^\pm \leq -d_{\text{min}}$, if $P$ would lie in the region labeled as not allowed in Figure 14, in which $\cos \theta < 0$, then there would be another point $P'$ of $A$ passing inside $T$, so that the value of the minimal distance would be strictly less than $d_{\text{min}}$.

First we prove that necessarily $e = 1$. Choose $e < 1$ and let $P, Q$ be two points on the two trajectories $A, A'$ attaining the minimal distance. By (14) we have

$$
\frac{\partial r}{\partial e} = \frac{\partial r}{\partial e} \hat{r},
$$

with $\frac{\partial r}{\partial e} \geq 0$, so that, using the continuous dependence on $e$ of the stationary points of $d^2$, the orbit distance increases by increasing $e$. In fact, by slightly increasing $e$, all the points on the trajectory $A$, in a neighborhood of $P$, will increase their distance from $A'$. The last statement is true because of relation
Figure 15: Section of the torus T containing the minimum points P, Q. The region not allowed for P is shaded.

(22). Note that this argument works also in case of more than one absolute minimum point.

We conclude that, if \( d_h = d_{\text{min}} \) and \( d_{\text{nod}} = -d_{\text{min}} \), then \( \frac{d}{d\theta} d^2_h \geq 0 \); thus the maximal value of \( d_{\text{min}} \) is attained for \( e = 1 \).

We now show that the maximal value of \( d_{\text{min}} \) in \( D_{h,2} \) is attained for \( \omega = \pi/2 \).

First we consider the case \( I = \pi/2 \). Since we are assuming the nodes of \( A \) are external to \( A' \), then the statements in steps 1 and 3 of Proposition 1 are fulfilled, so that, by steps 2 and 4 of the same proposition, we have \( \omega = \pi/2 \).

If \( I \neq \pi/2 \), we can write

\[
\frac{\partial d^2_h}{\partial \omega} = 4 \left( \frac{y(f_h)}{\cos I} x'(f'_h) - x(f_h) y'(f'_h) \cos I \right)
\]

\[
= 4x'(f'_h) y(f_h) \frac{\sin^2 I}{\cos I},
\]

where we have used \( x(f_h)y'(f'_h) = x'(f'_h)y(f_h) \), that follows from the second equation in (5). We show that, if \( d_h = d_{\text{min}} \), we have \( x'(f'_h)y(f_h) \geq 0 \), so that we can choose \( \omega = \pi/2 \).

By contradiction, assume \( x'(f'_h)y(f_h) < 0 \). Then we have

\[
(i) \quad \left\{ \begin{array}{cc} x'(f'_h) > 0 \\ y(f_h) < 0 \end{array} \right. \quad \text{or} \quad (ii) \quad \left\{ \begin{array}{cc} x'(f'_h) < 0 \\ y(f_h) > 0 \end{array} \right.
\]

(23)

If (i) holds then we can assume \( f_h + \omega \in (-\pi/2, 0) \), \( f'_h \in (-\pi/2, 0) \), in fact in the other cases we can select another value of \( f' \), keeping \( f = f_h \), and obtain a value of \( d \) smaller than \( d_h \). In a similar way, if (ii) holds we can assume \( f_h + \omega \in (\pi/2, \pi) \), \( f'_h \in (\pi/2, \pi) \).

In Table 1 we show a list of possible choices of \( f' \), depending on the values of \( (f_h, f'_h) \), that yield a value of \( d \) smaller than \( d_h = d(f_h, f'_h) \). These results can be checked using the relation

\[
d^2 = q^2 + r^2 - 2q'r \cos \alpha,
\]
\[
\begin{array}{|c|c|c|}
\hline
\text{range of } f_h + \omega & \text{range of } f_h' & \text{transformation} \\
\hline
(i) & (-\frac{\pi}{2}, 0) & (0, \frac{\pi}{2}) \\
 & (\pi, \frac{\pi}{2}) & (0, \frac{\pi}{2}) \\
 & (\pi, \frac{3\pi}{2}) & (0, \frac{3\pi}{2}) \\
\hline
(ii) & (0, \pi) & (\frac{\pi}{2}, \pi) \\
 & (\frac{\pi}{2}, 0) & (\pi, \frac{3\pi}{2}) \\
\hline
\end{array}
\]

Table 1: Transformations producing a lower value of \(d\).

where
\[
\cos \alpha = \cos(f + \omega) \cos f' + \sin(f + \omega) \sin f' \cos \theta
\]
and \(\alpha = \alpha(f, f')\) is the angle between \(r(f)\) and the position vector \(r'(f')\) of a point on \(A'\).

Now we exclude the case
\[
f_h + \omega \in (-\pi/2, 0), \quad f_h' \in (-\pi/2, 0).
\] (24)

Such a pair \((f_h, f_h')\) cannot attain the minimal value \(d_{\min}\) because in this case we would have (see Figure 16 (a))
\[
|d_{\text{nod}}^{+}| = \frac{2q}{1 + \cos \omega} - q' < \frac{2q}{1 + \cos f_h} - q' < \sqrt{\frac{4q^2}{(1 + \cos f_h)^2} + q'^2 - \frac{4qq'}{(1 + \cos f_h)} \cos \alpha_h} = d_{\min},
\]
where \(\alpha_h = \alpha(f_h, f_h')\). Note that we have used the relation \(q' < r(-\omega) < r(f_h)\), that yields from (24).

Finally we exclude the case
\[
f_h + \omega \in (\pi/2, \pi), \quad f_h' \in (\pi/2, \pi).
\] (25)

We consider two cases: \(f_h \leq \omega\) and \(f_h > \omega\). In the first case we have \(r(-\omega) \geq r(f_h)\). However, if we take \(-f_h\) in place of \(f_h\) we have \(r(f_h) = r(-f_h)\) and \(0 < z(-f_h) < z(f_h)\), so that the distance of the point corresponding to \(r(-f_h)\) from \(A'\) would be less than \(d_{\min}\), that is a contradiction (see Figure 16 (b)). In the second case we find that \(|d_{\text{nod}}^{+}|\) would be smaller than \(d_{\min}\). The geometric sketch of this last case is similar to the one in Figure 16 (a), here with \(z(f_h) > 0\); in fact also in this case we have \(q' < r(-\omega) < r(f_h)\).

We observe that we have not used \(e = 1\) in the proof that the maximal orbit distance in \(D_{5,2}\) is attained for \(\omega = \pi/2\). Therefore this result holds also for \(e < 1\).

We conclude this section by describing the possible values of \(d_{\min}\) as function of \(q\) only.
Figure 16: Geometry of the transformations decreasing the distance $d$. Here the two vertical sections of the torus $\mathbb{T}$ containing the relevant pairs of points $(P, Q)$ and $(\tilde{P}, \tilde{Q})$ are superposed. In (a) the pair $(\tilde{P}, \tilde{Q})$ corresponds to the ascending node. In (b) the point $\tilde{P}$ is obtained from $P$ by changing $f_h$ into $-f_h$. 
Corollary 1. Let $D_7 = \{ (e, I, \omega) : e \in [0, 1], I, \omega \in [0, \frac{\pi}{2}] \}$. For each $q \in (0, q_{\text{max}}]$ we have

$$\begin{align*}
\min_{(e, I, \omega) \in D_7} d_{\text{min}} &= \max \{ 0, q - q' \} \\
\max_{(e, I, \omega) \in D_7} d_{\text{min}} &= \max \{ q' - q, \delta(q) \}
\end{align*}$$

(26)

where $\delta(q)$ is the distance between $A'$ and $A$ with $e = 1$, $I = \omega = \frac{\pi}{2}$:

$$\delta(q) = \delta_\omega(q, \pi/2) = \sqrt{(\xi - q')^2 + \left(\frac{\xi^2 - 4q^2}{4q}\right)^2},$$

with $\xi = \xi(q)$ the unique real solution of $x^3 + 4q^2x - 8q'q^2 = 0$.

Proof. Lower bound. It follows immediately from Proposition 2.

Upper bound. From step 1 in Proposition 1 we can choose $I = \pi/2$. From step 2 in the same proposition the problem is reduced to the computation of the minimum of the distances between the nodes $Q_\pm$ of the Earth orbit and $A$, and we obtain either $\omega = 0$, or $\omega = \pi/2$. From Proposition 2 we know that we have either $e = 0$ or $e = 1$. Moreover, $e = 1$ can yield the maximal orbit distance only in case of external nodes, so that, by step 3 in Proposition 1 we obtain that in this case we have $\omega = \pi/2$.

Thus (26) holds, with

$$\delta(q) = \delta_\omega(q, \pi/2),$$

where $\delta_\omega$ is defined as in (16).

$\square$

4 The orbital distribution of the known NEAs

We use the bounds introduced in Section 3.2 to explain some selection effects in the orbital distribution of the known NEAs.

In Figure 17 we plot the pairs $(q, \omega)$ for the faint known NEAs, i.e. those with $H > 22$. In the same figure we draw the level curves of $(q, \omega) \mapsto \max_{D_7} d_{\text{min}}(q, \omega)$, and the curve $\gamma$ defined in (18). To plot Figure 17 we have used the invariance of $\max_{D_7} d_{\text{min}}$ under the symmetries

$$\omega \to \pi - \omega, \quad \omega \to \pi + \omega, \quad \omega \to 2\pi - \omega,$$

for $\omega \in [0, \pi/2]$.

We observe that these asteroids are concentrated close to the curve $\gamma$ (enhanced in Figure 17), representing values of $(q, \omega)$ such that $\max_{D_7} d_{\text{min}}(q, \omega)$ is small (see Figure 5). A reasonable explanation for that is the following: due to the geometric constraints, the faint asteroids with $(q, \omega)$ close to the curve $\gamma$ are easier to detect, whatever their values of $(e, I)$; on the contrary, the ones with $(q, \omega)$ far from $\gamma$ are not always detected. Moreover, the distribution of the values of $\omega$ for these asteroids can be assumed uniform. Therefore, the projection onto the plane $(q, \omega)$ yields this concentration effect.
Figure 17: Distribution of the known NEAs with $H > 22$ in the plane $(q, \omega)$.

Figure 18: Distribution of all the known NEAs in the plane $(q, e)$. The NEAs with $H > 22$ are plotted with darker gray.
In Figure 18 we draw the distribution of all the known NEAs in the plane $(q, e)$, together with the level curves of $(q, e) \mapsto \max_{D_3}(q, e)$. The asteroids with $H > 22$ are plotted with darker gray. We can observe a concentration of these faint NEAs around $q = q'$. However, this case is different from the previous plot, since the distribution of the values of $e$ cannot be assumed uniform. The enhanced curve in Figure 18 describes the boundary with the inner-Earth asteroids, i.e. the curve $q(1 + e)/(1 - e) = q'$, so that we have only a few NEAs below it. On the other hand, from Figures 7, 8 we can see that the value of the minimal distance $\min_{D_3} d_{\min}$ steeply increases as we go below this curve. The lack of asteroids in the upper part of Figure 18 cannot be explained in terms of orbit distance. The straight lines passing through $(q, e) = (0, 1)$ correspond to different periods (labeled in yrs) for the asteroid orbit. Thus, waiting for future survey operations, some asteroids which have not been discovered yet should appear in this portion of the plane $(q, e)$. This time limitation is unavoidable.

In Figure 19 we draw the distribution of all the known NEAs in the plane $(q, I)$, together with the level curves of $(q, I) \mapsto \max_{D_5}(q, I)$. The asteroids with $H > 22$ are plotted with darker gray. To plot this figure we have used the invariance of $\max_{D_5} d_{\min}$ under the symmetry $I \mapsto \pi - I$ for $I \in [0, \pi/2]$. Here we can only observe a concentration of very faint NEAs around $q = q' = 1$ au, with low inclinations. However, we recall that the distribution of the values of $I$ is very far from uniform.

In Figure 20 we draw the distribution of the known NEAs in the plane $(q, d_{\min})$. The maximal value of $d_{\min}$ found for the 9220 known NEA orbits (to the date of 21/10/2012) is 0.7036 au, attained by asteroid 2010 KY₁₂₇.
Figure 20: Distribution of the known NEAs in the plane \((q, d_{\text{min}})\). The shaded region represents the possible values of \(d_{\text{min}}\) as function of \(q\). The NEAs with \(H > 22\) are plotted with darker gray.

The V-shaped structure appearing in Figure 20 corresponds to the graph of \(q \rightarrow |q - q'|\), and can be understood by looking at the projection in the direction of \(\omega\) of the minimal and maximal orbit distance surfaces \(\min_D, d_{\text{min}}, \max_D, d_{\text{min}}\) defined in Proposition 2 (see Figure 5).

5 The threshold of the NEA class

In Figure 21 we show how the maximal value of \(d_{\text{min}}\) varies by a small change of the threshold value of \(q_{\text{max}}\), defining the NEA class. Note that the value \(q_{\text{max}} = 1.3\) au, introduced in [13], is very peculiar. Actually if we choose e.g. \(q_{\text{max}} = 1.299\) au the maximal value of \(d_{\text{min}}\) becomes 1 au; this value is obtained by a Sun-grazing trajectory, perpendicular to \(\mathcal{A}'\).

5.1 Apparent magnitude at the maximal orbit distance

Every survey has a limiting apparent magnitude: this defines a threshold for the size of the NEAs that can be discovered. In fact the apparent magnitude \(H_{\text{app}}\) is related to the absolute magnitude \(H_{\text{abs}}\), and through the latter we obtain an estimate of the asteroid diameter. We use an approximated formula for this relation, taken from [3], that we recall below. Given the heliocentric and geocentric position of the asteroid \((\mathcal{X}, \mathcal{X}' - \mathcal{X}'')\) respectively we can compute the phase angle \(\beta\) (i.e. the angle Sun-Asteroid-Earth). If we fix a value for the
Figure 21: The maximal orbit distance as function of $q_{\text{max}}$.

slope parameter $G$ (depending on the albedo of the asteroid surface) we have

$$H_{\text{app}} = H_{\text{abs}} + 5 \log_{10}(r \rho) - 2.5 \log_{10}((1 - G) \phi_1 + G \phi_2),$$

where

$$r = |\mathcal{X}|, \quad \rho = |\mathcal{X} - \mathcal{X}'|,$$

$$\phi_j = \exp\left[-a_j \left(\tan \left(\frac{\beta}{2}\right)\right)^{b_j}\right], \quad j = 1, 2,$$

$$a_1 = 3.33, \quad a_2 = 1.87, \quad b_1 = 0.63, \quad b_2 = 1.22.$$  

We consider an object on the parabolic trajectory $\mathcal{P}$, defined in step 3 of Proposition 1. If we set $G = 0.15$, which corresponds to a typical value for asteroids, and assume this object is observed from the Earth when the two bodies are at the minimal distance points, then we have

$$\mathcal{X} \approx (1.5, 0, 0.867) \text{ au}, \quad \mathcal{X}' \approx (1.0, 0, 0) \text{ au},$$

so that

$$r \approx 1.73 \text{ au}, \quad \rho \approx 1.0011 \text{ au}.$$  

In conclusion we obtain

$$\beta \approx 30^\circ, \quad H_{\text{app}} \approx H_{\text{abs}} + 2.495.$$  

6 Conclusions

We have proven certain geometric properties of confocal conics and shown that they can explain some selection effects in the orbital distribution of the known NEAs. These asteroids have been detected for the large majority with ground-based observations. The results of this paper are relevant to plan future surveys,
with space-based observations: in fact the optimal bounds in Propositions 2, 3, 4 hold for arbitrary values of $q' > 0$. Thus, for example, placing a space telescope in a circular orbit with radius equal to the average semimajor axis of Venus ($q' \approx 0.72$ au) we should discover several faint NEAs. The suggestion of placing telescopes in satellite orbits much interior to the Earth, to increase the NEA discovery, can be found also in [10], [12]. In Figure 22 we show the $\gamma$ curves corresponding to different choices of $q'$, with the background of the known fainter NEAs ($H > 22$).

Finally we note that the surfaces of minimal and maximal distance can be useful in a stress test for a program computing the orbit distance. In fact one could perform a very large number of orbit distance computations, with one circular orbit, and check if the computed values lie within the bounds of Propositions 2, 3, 4.

References


A.1 A result by Apollonius

The distance between a conic section and a point in the plane has been studied very long ago (≈ 200 BC) by Apollonius of Perga, see [9]. Here we recall some of his results in modern mathematical language (see [8]). Choose rectangular coordinates \((x, y)\) in the plane and consider the ellipse \(\mathcal{A}\), defined by the equation

\[
\frac{[(1-e)x+eq]^2}{q^2} + \frac{y^2(1-e)}{q^2(1+e)} = 1 ,
\]

where \(q, e\) are the pericentre distance and the eccentricity of \(\mathcal{A}\), respectively.

Moreover, let \(Q\) be the point with coordinates \((x_0, y_0) = (q' \cos \omega, q' \sin \omega)\). We want to compute the minimum value of the distance \(d_Q\) between \(Q\) and a point on \(\mathcal{A}\), which can be parametrized by the true anomaly \(f\), obtaining in
Figure 23: We draw the two branches of Apollonius’ hyperbola $\mathcal{H}$ in a case with 4 intersections with the ellipse $\mathcal{A}$. The star-shaped figure is the graph of the evolute of $\mathcal{A}$.

This way a function $f \mapsto d_Q(f)$. We can consider all the normals from $Q$ to the ellipse $\mathcal{A}$: the intersections of these normals with $\mathcal{A}$ correspond to the stationary points of $f \mapsto d_Q(f)$.

The hyperbola $\mathcal{H}$, defined by

$$[(1-e)x + eq][ye^2 + y0(1-e^2)] - [(1-e)x0 + eq] = 0,$$

has the asymptotes parallel to the coordinate axes and one branch passing through $Q$ and the centre $O$ of the ellipse (see Figure 23). Moreover, $\mathcal{H}$ intersect $\mathcal{A}$ just in the stationary points of $d_Q$. It turns out the $d_Q$ has at most 4 stationary points, and at most 2 minima. The case with two minima occurs when both branches of $\mathcal{H}$ intersect $\mathcal{A}$ so that, due to the alternation of maxima and minima of $d_Q(f)$, there is 1 minimum point per branch. Apollonius also introduced the evolute of $\mathcal{A}$, which is a curve with parametric equation

$$x(\phi) = \frac{qe^2}{1-e}\cos^3 \phi, \quad y(\phi) = -\frac{qe^2}{(1-e)\sqrt{1-e^2}}\sin^3 \phi,$$

with $\phi \in S^1$. The evolute of the ellipse is a closed, star-shaped curve. Moreover, both branches of $\mathcal{H}$ intersect $\mathcal{A}$ if and only if $Q$ lies inside the graph of the evolute (like in Figure 23).

In step 2 of Proposition 1 we study the distance between $Q$ and $\mathcal{A}$ as the argument of perihelion $\omega$ varies in $[0, \pi]$. To this aim we can of course leave $\mathcal{A}$ fixed and rotate by $-\omega$ the point $Q$ around the focus $O$, see Figure 23.

From these geometric results it is easy to deduce that an exchange of role between local minima as absolute minimum of $d_Q(f)$ can occur only for $\omega = 0, \pi$. 

30
A similar statement holds also if $A$ is a parabola. If the equation of $A$ is
\[ x - \frac{4q^2 - y^2}{4q} = 0, \]
then the normals from $Q$ to $A$ are at most $3$, and they are found by looking for the intersections between $A$ and the hyperbola
\[ xy - (2q + x_0)y + 2qy_0 = 0. \]

A.2 The curve $\gamma$

We describe the procedure to compute the curve $\gamma$ in the plane $(q, \omega)$, given in equation (18).

System (17) can be written as $p_1 = p_2 = 0$, where
\[
\begin{align*}
p_1(x) &= x^4 + 8q(q' \cos \omega + q)x^2 - 32q^2q' \sin \omega x \\
p_2(x) &= x^3 + 4q(q' \cos \omega + q)x - 8q^2q' \sin \omega.
\end{align*}
\]
are polynomials in the $x$ variable. By computing the resultant $\text{Res}(p_1, p_2, x)$ of $p_1, p_2$ with respect to $x$, the dependence on $\sin \omega$ disappears. Setting $y = \cos \omega$ we obtain
\[
\begin{align*}
\text{Res}(p_1, p_2, x) - 4096q^7(q' - 1) &= 2q^4 + 2q'^3(7y - 5)q^3 - 2q^2(3y + 22)(y - 1)q^2 + q'^3(y^3 + 13y^2 + 9y - 27)q - 2q^3q'^4 \\
&= q'^3(-2q' + q)y^3 - q'^2q(6q - 13q')y^2 + q'(9q'^2 - 38q'q + 14q^2)y + q(2q^3 - 27q'^3 + 44q^2q - 10q^2q').
\end{align*}
\]
The factor $q^7(y - 1)$ appears because $\delta_\omega$ is equal to $|q - q'|$ for $y = 1$ or $q = 0$; however this factor must be eliminated to obtain the algebraic expression for $\gamma$.

A.3 Computation of $\delta_e(q, e)$

We compute the orbit distance between $A'$ and a trajectory $A$ with $I = \pi/2$, $\omega = \pi/2$, as a function of $(q, e)$.

From step 2 of Proposition 1 we have to compute the distance of a nodal point $Q_\pm \in A'$ from the trajectory $A$. To this aim we can consider only the portion of $A$ which is the graph of the function
\[
x \mapsto z(x) = \frac{qe}{1 - e} - \sqrt{\frac{q^2}{(1 - e)^2} - \frac{x^2}{1 - e^2}},
\]
where
\[
x \in \left[-q\sqrt{\frac{1 + e}{1 - e}}, q\sqrt{\frac{1 + e}{1 - e}}\right].
\]
We introduce
\[ D^2(x) = (x - q')^2 + z^2(x) \]
and consider the stationarity condition
\[ (D^2)' = 0. \] (27)
From (27) we obtain the polynomial equation
\[
\begin{align*}
e^4x^4 + 2q'e^2(1 - e^2)x^3 + (1 + e)^2(q^2(e - 1)^2 + q^2e^2)x^2 \\
-2q'e^2q^2(1 + e)^2x - q^2q^2(1 - e^2)(1 + e)^2 = 0.
\end{align*}
\] (28)
By Descartes’ rule of signs, equation (28) has only one real positive solution \( \bar{x} \), that corresponds to a component of an absolute minimum point.

A.4 Computation of \( \delta_I(q, I) \)

We compute the orbit distance between \( A' \) and a trajectory \( A \) with \( e = 1, \omega = \pi/2 \), as a function of \((q, I)\).

From the proof of Proposition 4 we know that we are interested only in the case with \( I \neq 0 \) and the nodes of \( A \) external to \( A' \). We also know that \( y'_{\min} < 0 \) (it follows from the proof that \( \partial d_h^2 / \partial \omega \geq 0 \) if \( d_h = d_{\min} \)).

We use the following parametrization:
\[
x' = -q' \sin \eta', \quad y' = q' \cos \eta', \quad \eta' = f' - \pi/2; \\
y = \zeta \cos I, \quad z = \zeta \sin I, \quad \zeta = q - \frac{x^2}{4q}.
\]
Then we set
\[
d^2(x, \eta') = (x - x')^2 + (y - y')^2 + z^2
\]
and compute the stationary points of \( d^2 \):
\[
\begin{align*}
4q^2 \frac{\partial d^2}{\partial x} &= 4q^2x + 8q^2q' \sin \eta' + 4qq' \cos I x \cos \eta' + x^3 = 0, \\
2q^3 \frac{\partial d^2}{q' \partial \eta'} &= 4qx \cos \eta' - \cos I x^2 \sin \eta' + 4q^2 \cos I \sin \eta' = 0.
\end{align*}
\]
Apply the coordinate change
\[ s = \tan(\eta'/2), \] (29)
so that
\[
\cos \eta' = \frac{1 - s^2}{1 + s^2}, \quad \sin \eta' = \frac{2s}{1 + s^2}.
\]
Note that, since \( \eta' = f' + \pi/2 \), the point corresponding to \( \eta' = \pi \), which is sent to infinity by (29), cannot correspond to the \( f' \) component of the absolute minimum point, not even for \( I = \pi/2 \).
We obtain the polynomial equations
\[
\begin{align*}
p_1(x,s) &= 2qx(1 - s^2) - s \cos I(x^2 - 4q^2) = 0, \\
p_2(x,s) &= (x^2 + 4q^2)x(1 + s^2) + 16q^2q's + 4qq' \cos Ix(1 - s^2) = 0.
\end{align*}
\]

The resultant of \( p_1, p_2 \) with respect to \( s \) is the polynomial
\[
\text{res}(x) = x^2[\cos^2 Ix^8 + 16q^2x^6 - 16q^2(2q^2(\cos^2 I - 4) + q'^2 \cos^4 I)x^4 + 128q^4(2q^2 + q'^2 \cos^2 I(\cos^2 I - 2))x^2 - 256q^6(q'^2(\cos^2 I - 2)^2 - q^2 \cos^2 I)] .
\]

For each root \( \bar{x} \) of \( \text{res}(x) \) we search for the value \( \bar{s} \) of \( s \) corresponding to a stationary point:
\[
2qx\bar{s}^2 + \cos I(x^2 - 4q^2)\bar{s} - 2qx = 0. \tag{30}
\]

Equation (30) has the roots
\[
\bar{s}_{1,2} = \frac{- \cos I(x^2 - 4q^2) \pm \sqrt{\Delta}}{4qx},
\]
with
\[
\Delta = \cos^2 I(x^4 - 8q^2x^2 + 16q^4) + 16q^6x^2 .
\]

To search for \( d_{\text{min}} \), the absolute minimum \( d(x, \eta') \), we can restrict to
\[
\eta' \in [-\pi/2, 0], \quad x \geq 0 .
\]