An elementary quantum correction to the Child–Langmuir solution for the vacuum diode

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December 11, 2005

1 Introduction

C. D. Child ([1]) and I. Langmuir ([2]) gave a formula describing the current density in a vacuum diode as a function of the difference of potential between the cathode and anode. First we review Child–Langmuir's classical solution (C-L solution), then we investigate in an elementary way the changes to be done to their formula when a very rarefied monoatomic gas (e.g. mercury vapor) is present in the diode. These changes take into account the anelastic collisions of the electrons, emitted by the cathode and accelerated by the electric field, against the atoms of the gas.

We assume that the electrons give entirely their kinetic energy to the atoms of the gas as soon as such energy reaches a critical threshold $\mathcal{K}$; at the same time we keep the continuity of the electric field and of the potential.

We shall prove some properties of the new law for the current density, in particular we shall show that it keeps the same monotonicity as in Child–Langmuir's classical law.

2 The classical C-L solution

Let us consider a vacuum diode, that is a device composed by a cathode and an anode: this device can be thought as composed by a hot filament (the cathode) surrounded by a metallic cylinder (the anode), which is positively charged (see Figure 1A).

By the effect of thermionic emission some negatively charged particles leave the cathode and, due to the presence of the electric field, can reach the anode.

For simplicity, as in Langmuir's original paper, we shall assume that the cathode and the anode are infinitely large, so that we can approximate them with two parallel planes at distance L, as in Figure 1B. Let m, v, e be respectively the mass, the size of the velocity and the absolute value of the charge of an electron. Let $\phi(x)$ be the electric potential, as a function of the distance x from

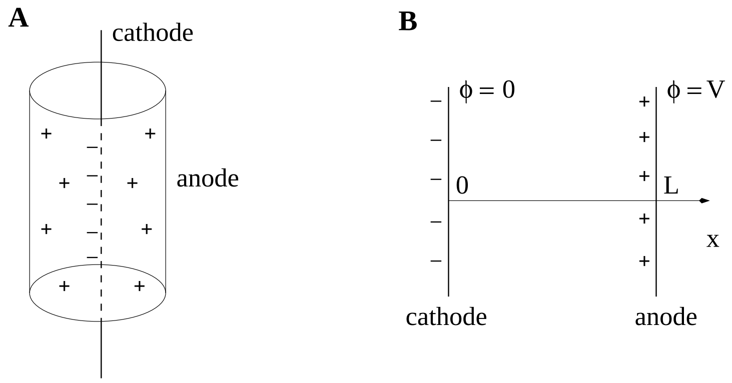

Figure 1: Representation of a diode (A) and its approximation with two infinite parallel planes (B).

the cathode. We shall assume that the potential difference between the cathode and the anode is V and that the potential is zero on the cathode:

$$\phi(0) = 0\,, \qquad \phi(L) = V\ . \tag{1}$$

Let ρ be the charge density and J the current density, so that relation

$$J = -\rho v \tag{2}$$

holds. From Poisson's equation

$$-\phi'' = \frac{\rho}{\epsilon_0}\,, \tag{3}$$

where ϵ_0 is the electric permittivity in the vacuum, and from relation (2) we obtain

$$\phi'' = \frac{J}{\epsilon_0 v}\ .$$

We compute $v = v(\phi)$ using the energy conservation law

$$\frac{1}{2}mv^2 - e\phi = \mathcal{E}\,,$$

where we can set $\mathcal{E} = 0$, that implies $\phi \geq 0$. Finally we notice that J is constant, due the current conservation law. We introduce the positive constant a, such that

$$a^2 = \frac{J}{\epsilon_0}\sqrt{\frac{m}{2e}}\ . \tag{4}$$

We obtain in this way the following Cauchy problem with limit conditions for the potential ϕ:

$$\begin{cases} \phi'' = a^2\phi^{-1/2} \\ \phi(0^+) = 0 \\ \phi'(0^+) = 0 \end{cases} \ . \tag{5}$$

Multiplying by ϕ' and integrating between $\eta > 0$ and x, with $0 < \eta < x < L$, we obtain the first order ODE[1]

$$\phi'(x) = \sqrt{\phi'^2(\eta) + 4a^2\left(\sqrt{\phi(x)} - \sqrt{\phi(\eta)}\right)} \ . \tag{6}$$

Passing to the limit for $\eta \to 0^+$ and using the asymptotic conditions in (5) we obtain

$$\begin{cases} \phi'(x) = 2a(\phi(x))^{1/4} \\ \phi(0^+) = 0 \end{cases} . \tag{7}$$

The solution to problem (7), and then to (5), is

$$\phi(x) = \left(\frac{3}{2}ax\right)^{4/3} . \tag{8}$$

By varying the end potential V we obtain different values of the electric current: using relations (1) and (4) we can write

$$J(V) = \epsilon_0 \sqrt{\frac{2e}{m}} \frac{4}{9L^2} V^{3/2} \tag{9}$$

that is the classical C-L solution.

3 Application of a quantum correction

We assume that a very rarefied monoatomic gas is present in the diode and that the electrons leaving the cathode that collide against the atoms of the gas give their whole kinetic energy to these atoms when it reaches the critical threshold $\mathcal{K}$, that we assume to be known. In this section we shall study the case in which the threshold is reached only once on the way from the cathode to the anode.

In correspondence of the critical value $\mathcal{K}$ there will be a value $\xi \in]0, L[$ (to be determined) of the x variable such that the kinetic energy of an electron $mv^2/2$ vanishes abruptly, with

$$\lim_{x \to \xi^-} \frac{1}{2}mv^2 = \mathcal{K} ; \qquad \lim_{x \to \xi^+} \frac{1}{2}mv^2 = 0 \ .$$

We shall call ϕ_0 and ϕ_1 the restrictions of the electric potential to the intervals $I_0 =]0, \xi[$ and $I_1 =]\xi, L]$ respectively and we impose to the solution the following regularity conditions

$$\begin{cases} \phi_0(\xi^-) = \phi_1(\xi^+) \\ \phi_0'(\xi^-) = \phi_1'(\xi^+) \end{cases} . \tag{10}$$

Thus we have two different energy conservation laws on the two intervals I_0, I_1:

$$\frac{1}{2}mv^2 - e\phi_h = \mathcal{E}_h, \qquad h = 0, 1 \tag{11}$$

[1] In this procedure we have chosen the plus sign for the derivative of ϕ: this makes strictly increasing the electric potential and implies $\phi \geq 0$, according to the energy conservation law.

where $\mathcal{E}_0 = 0$, and $\mathcal{E}_1$ is to be determined. Passing to the limit for $x \to \xi^-$ and for $x \to \xi^+$ in the two conservation laws (11) and using the regularity conditions (10) we obtain

$$\mathcal{E}_1 = -\mathcal{K}.$$

In the interval I_0 the electric potential ϕ_0 is solution of the Cauchy problem (5), where in this case we use the dielectric permittivity ϵ in place of ϵ_0. Thus ϕ_0 has the same expression as ϕ in (8): from this expression and from the conservation of the energy in I_0 we can compute ξ by passing to the limit for $x \to \xi^-$:

$$\xi = \frac{2}{3a}\left(\frac{\mathcal{K}}{e}\right)^{3/4} .$$

Let us introduce the modified electric potential

$$\psi_1(x) = \phi_1(x) + \frac{\mathcal{E}_1}{e} = \phi_1(x) - \frac{\mathcal{K}}{e} .$$

We obtain the Cauchy problem

$$\begin{cases} \psi_1'' = a^2 \psi_1^{-1/2} \\ \psi_1(\xi^+) = 0 \\ \psi_1'(\xi^+) = 2a\left(\frac{\mathcal{K}}{e}\right)^{1/4} \end{cases} \quad \text{for } x \in I_1, \tag{12}$$

where the (asymptotic) initial values are obtained by applying the regularity conditions (10).

Integrating the differential equation in (12) between ξ and x and evaluating the result for $x = L$ we obtain

$$L - \frac{2}{3a}\left(\frac{\mathcal{K}}{e}\right)^{3/4} = \frac{2}{3a}\left[\sqrt{\sqrt{\frac{\mathcal{K}}{e}} + \sqrt{V - \frac{\mathcal{K}}{e}}}\left(\sqrt{V - \frac{\mathcal{K}}{e}} - 2\sqrt{\frac{\mathcal{K}}{e}}\right) + 2\left(\frac{\mathcal{K}}{e}\right)^{3/4}\right] \tag{13}$$

so that

$$a(V) = \frac{2}{3L}\left[3\left(\frac{\mathcal{K}}{e}\right)^{3/4} + \sqrt{\sqrt{\frac{\mathcal{K}}{e}} + \sqrt{V - \frac{\mathcal{K}}{e}}}\left(\sqrt{V - \frac{\mathcal{K}}{e}} - 2\sqrt{\frac{\mathcal{K}}{e}}\right)\right] ; \tag{14}$$

using relation (4) the law for the current density as a function of V becomes

$$J_1(V) = \epsilon\sqrt{\frac{2e}{m}}\frac{4}{9L^2}\left[3\left(\frac{\mathcal{K}}{e}\right)^{3/4} + \sqrt{\sqrt{\frac{\mathcal{K}}{e}} + \sqrt{V - \frac{\mathcal{K}}{e}}}\left(\sqrt{V - \frac{\mathcal{K}}{e}} - 2\sqrt{\frac{\mathcal{K}}{e}}\right)\right]^2 . \tag{15}$$

Remark: The function J_1 is defined for $V \geq \frac{\mathcal{K}}{e}$ and for $\mathcal{K} = 0$ we have again relation (9).

By defining $J_0(V) = J(V)$, given in (9), and separately considering the two different behaviors of the current density for $V < \mathcal{K}/e$ and for $V > \mathcal{K}/e$ we can write the *modified C-L law*

$$J(V) = \begin{cases} J_0(V) & \text{for } V \in [0, \frac{\mathcal{K}}{e}[\\ J_1(V) & \text{for } V \geq \frac{\mathcal{K}}{e} \end{cases} \tag{16}$$

In Figure 2 we compare the modified C-L law (16) with the classical one. We plot the graphic of the current density J as a function of the potential difference V: the hyphened curve represents the current in the classical C-L law, while the continuous curve is the current given in (16). Taking into account anelastic collisions of the electrons against the gas atoms the current of course decreases.

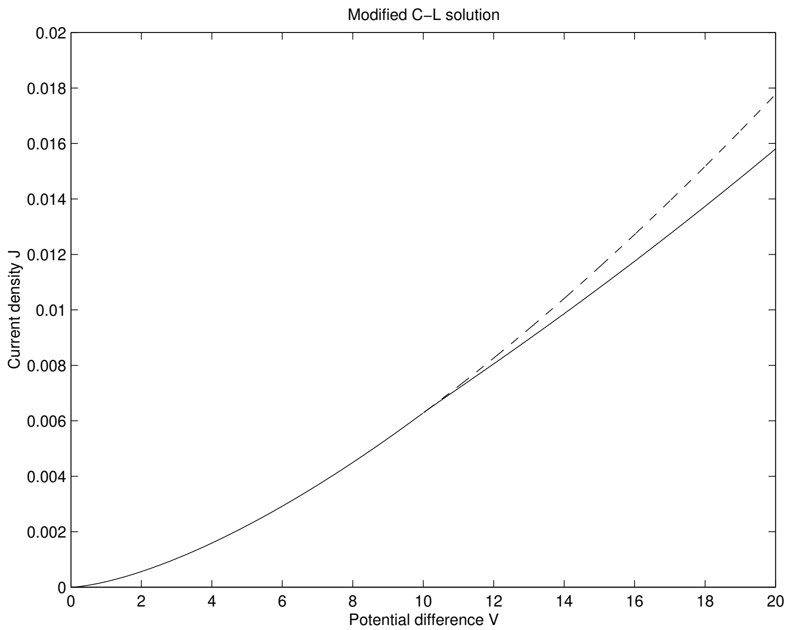

Figure 2: Current density as a function of the potential difference: the trend of the C-L law is compared with the one given in (16).

3.1 Monotonicity of $J(V)$

The function $J_0(V)$ in (9) is monotone increasing: we investigate if the addition of the quantum correction can change the monotonicity properties of the current as function of the electric potential difference. To this aim we compute

$$\frac{dJ_1}{dV}(V) = \frac{2\epsilon}{3L^2}\sqrt{\frac{2e}{m}}\, A_{\mathcal{K}/e}\, B_{\mathcal{K}/e} \tag{17}$$

where

$$A_{\mathcal{K}/e}(V) = \left[3 \left(\frac{\mathcal{K}}{e} \right)^{3/4} + \sqrt{\sqrt{\frac{\mathcal{K}}{e}} + \sqrt{V - \frac{\mathcal{K}}{e}}} \left(\sqrt{V - \frac{\mathcal{K}}{e}} - 2\sqrt{\frac{\mathcal{K}}{e}} \right) \right]$$

and

$$B_{\mathcal{K}/e}(V) = \frac{1}{\sqrt{\sqrt{\frac{\mathcal{K}}{e}} + \sqrt{V - \frac{\mathcal{K}}{e}}}}$$

is a positive factor. The factor $A_{\mathcal{K}/e}(V)$ can be negative only if

$$\sqrt{V - \frac{\mathcal{K}}{e}} - 2\sqrt{\frac{\mathcal{K}}{e}} < 0 \,,$$

that is only if $V < 5\mathcal{K}/e$. For $V \in [\mathcal{K}/e, 5\mathcal{K}/e]$ we square both the (positive) sides of relation

$$3 \left(\frac{\mathcal{K}}{e} \right)^{3/4} < \sqrt{\sqrt{\frac{\mathcal{K}}{e}} + \sqrt{V - \frac{\mathcal{K}}{e}}} \left(2\sqrt{\frac{\mathcal{K}}{e}} - \sqrt{V - \frac{\mathcal{K}}{e}} \right)$$

and we find

$$9 \left(\frac{\mathcal{K}}{e} \right)^{3/2} < f_{\mathcal{K}/e}(V) \tag{18}$$

with

$$f_{\mathcal{K}/e}(V) = \left(\sqrt{\frac{\mathcal{K}}{e}} + \sqrt{V - \frac{\mathcal{K}}{e}} \right) \left[2\sqrt{\frac{\mathcal{K}}{e}} - \sqrt{V - \frac{\mathcal{K}}{e}} \right]^2$$

a non-increasing monotonic function in the interval $[\mathcal{K}/e, 5\mathcal{K}/e]$, with maximal value $4\,(\mathcal{K}/e)^{3/2}$. Thus relation (18) cannot hold for any value of V in this interval and the function $J_1(V)$ is monotone increasing.

3.2 Regularity of $J(V)$

We observe that the function $J(V)$ defined in (16) is continuous, in fact a simple computation gives

$$J_1 \left(\frac{\mathcal{K}^+}{e} \right) = \epsilon \sqrt{\frac{2e}{m}} \frac{4}{9L^2} \left(\frac{\mathcal{K}}{e} \right)^{3/2} = J_0 \left(\frac{\mathcal{K}^-}{e} \right) \,.$$

We can even show that it is $\mathcal{C}^1$, in fact from (17) we obtain

$$\frac{dJ_1}{dV} \left(\frac{\mathcal{K}^+}{e} \right) = \epsilon \sqrt{\frac{2e}{m}} \frac{2}{3L^2} \left(\frac{\mathcal{K}}{e} \right)^{1/2} = \frac{dJ_0}{dV} \left(\frac{\mathcal{K}^-}{e} \right) \,.$$

References

[1] CHILD, C. D., 1911: *'Discharge from hot CaO'*, Phys. Rev, **32**, p. 492

[2] LANGMUIR, I., 1913: *'The Effect of Space Charge and Residual Gases on Thermionic Currents in High Vacuum'*, Phys. Rev, **2**, p. 450