Theoretical and computational aspects of collision singularities in the N-body problem

Contents

Chapter 1

Introduction

1.1 Description of the results

We present some results concerning the N-body problem. The problems that we consider are deeply different and the mathematical tools used to deal with them come from several branches of Mathematics: Perturbation Theory, Analytical Mechanics, Mathematical Analysis, Algebraic Geometry and Numerical Analysis. These results have however a common feature: they are solutions of problems arising from the presence of collision singularities.

In Chapter 2 we describe the main features concerning the N-body problem, starting from its definition. We also speak about the regularization of the collisions in a system with two particles.

In Chapter 3 we review some results about the averaging principle, that can be used to study the qualitative behavior of systems of Ordinary Differential Equations (ODEs), and we apply this principle to the computation of the secular evolution of Main Belt Asteroids (MBAs).

After these preliminary chapters we introduce the original part of this work.

In Chapters 4, 5 we describe a generalization of the averaging principle in order to compute the secular evolution of Near Earth Asteroids (NEAs): as these asteroids can cross the orbit of the Earth, the classical averaging principle cannot be applied because of the singularities coming from the existence of collision solutions when there is an intersection between the orbits. This generalization of the averaging principle allows to define piecewise smooth averaged solutions: it has been developed using a model with the planets on circular coplanar orbits; in the following we shall refer to this as to the *circular coplanar case*.

In Chapter 6 we prove a stability property of the averaged evolution of NEAs in the framework of the circular coplanar problem: for each value of the osculating semimajor axis and inclination of a NEA and for each number of perturbing planets in the model (with arbitrary masses and orbital radii) there always exists a region in the phase space where the motion is stable and is not affected by crossings with the orbit of any planet.

In Chapter 7 we describe the secular evolution theory and the method to compute actually the averaged evolution of planet crossing orbits. This theory is used to compute proper elements for NEAs, that are quantities that can be considered approximatively constant during the real motion (they are actually constant during the averaged motion). These elements can be used for several purpose, including the study of the possibility of collision with the Earth and the study of dynamical mechanisms, such as secular resonances.

The reliability of the secular evolution theory is examined in Chapter 8, where we show the secular evolution of a sample of NEAs, with very different behaviors, compared with the results of pure numerical integration starting from initial conditions with the planets on circular coplanar orbits and sampling different choices for the initial phase of the asteroid on its own orbit.

The generalized averaging principle is then extended in Chapter 9, where we include the real eccentricity and inclination of the planets in the model. This allows to compute more reliable crossing times for the averaged motion.

In Chapter 10 we introduce an algebraic method for the computation of all the stationary points (or critical points) of the squared distance d^2 between a point on one ellipse and a point on a second ellipse with a focus in common with the first. Some applications of the algorithm to the orbits of NEAs and the orbit of the Earth are shown. This algorithm is based on the use of the Fast Fourier Transform to obtain the coefficients of the resultant of the two bivariate components of the gradient of d^2 with respect to one variable. We also prove an upper bound to the total number of stationary points that we have to expect in this problem and we show some examples of orbital configurations with a high number of critical points.

In Chapter 10 we study the limit trajectories of a generalized two body problem: we consider a potential of the form $1/(r^2+\varepsilon^2)^{\alpha/2}$ (r is the relative distance between the two bodies and ε is a small real parameter) and we compute the limits for ε and ν (a quantity proportional to the norm of the angular momentum) that go to zero. Such limits do not exists for each exponent α and in particular they do not exist for $\alpha = 1$, corresponding to the Kepler problem if $\varepsilon = 0$. In this chapter the discontinuities of this generalized two body problem have been put in evidence: we can obtain quite different limit trajectories by varying the dependences between ε and ν, from the bounce (corresponding to the trajectories of the Levi-Civita regularization) to a spiral motion, turning around the collision infinitely many times. These results are probably related to the regularizability of the vector field in the sense of Easton (in which the regularized solutions are defined on the basis of the continuity property of the flow).

1.2 Acknowledgments

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sity of Pisa, Computer Algebra Group), P. Michel (*Nice Observatory*).

Chapter 2

The N-body problem

The definition of the problem and the classical results about it are given in a general framework following [16].

2.1 Definition of the N-body problem

Given N points with masses $m_1, \dots, m_N$, moving in the Euclidean space $\mathbb{R}^3$, we denote by

$$q_i = (q_{i,1}, q_{i,2}, q_{i,3}) \in \mathbb{R}^3$$

the position vector of the mass m_i for each $i = 1 \dots N$ and we set

$$q = (q_1, \dots, q_N) \ \in \mathbb{R}^{3N} \ ;$$

the vector q represents the *configuration* of the N points.

Given a real number $\alpha > 0$, we define the *potential* (or *force function*) of the system as

$$U(q) = \sum_{1 \le i < j \le N} G m_i m_j ||q_i - q_j||^{-\alpha} \tag{2.1}$$

where G is Newton's gravitational constant and $||\cdot||$ is the Euclidean norm in $\mathbb{R}^3$. We also define the set of *collision configurations* as

$$\Delta = \bigcup_{1 \le i < j \le N} \left\{ q \in \mathbb{R}^{3N} : q_i = q_j \right\} \ .$$

The potential energy of the system is $-U$, so that the equations of motion are

$$\ddot{q} = M^{-1} \nabla U(q) \ ,$$

where M is the diagonal matrix

$$M = \begin{bmatrix} M_1 & 0 & 0 & \dots & 0 \\ 0 & M_2 & 0 & \dots & 0 \\ 0 & 0 & M_3 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & M_N \end{bmatrix}$$

with

$$M_i = \begin{bmatrix} m_i & 0 & 0 \\ 0 & m_i & 0 \\ 0 & 0 & m_i \end{bmatrix} \qquad \text{for } i = 1 \dots N \ .$$

We can also write the equations of motion as a first order system using the Hamiltonian formalism:

$$\begin{cases} \dot{q} = M^{-1} p \\ \dot{p} = \nabla U(q) \ , \end{cases} \tag{2.2}$$

where $p = M\dot{q}$ is the momentum, that is the vector variable conjugate to q.

Remark 1. If we set $\alpha = 1$ in (2.1) we obtain the Newtonian potential.

2.2 The nature of the singularities

If we select *initial conditions* $(q_0, p_0) \in \left\{\mathbb{R}^{3N} \setminus \Delta\right\} \times \mathbb{R}^{3N}$ at time $t = 0$, then the theorem of local existence and uniqueness of the solutions of Ordinary Differential Equations can be applied to system (2.2). We have a unique real analytic solution defined on a maximal interval $]t_-, t_+[$.

We give the following definitions:

1. if $t_+ < +\infty$ then we say that the solution is *singular* (or that *ends into a singularity*) and we say that t_+ is a singularity;
2. if $t_- > -\infty$ then we say that the solution is *singular* (or that *begins in a singularity*) and we say that t_- is a singularity;
3. if $t_- = -\infty$ and $t_+ = +\infty$ then the solution is said to be *regular*.

Due to the symmetry of the problem we can reduce ourselves to the study of the problem on the interval $]t^*, 0]$ where $t^* \in \mathbb{R}^-$ (negative real numbers) is a singularity.

We define the function

$$\rho : \mathbb{R}^{3N} \to [0, +\infty[\,; \qquad \rho(q) = \min_{1 \le i < j \le N} q_{ij}\,;$$

where $q_{ij} = ||q_i - q_j||$.

The following result holds:

Theorem 1. *(Painlevè) If (q,p) is an analytic solution of the equations (2.2) defined on a maximal interval $]t^*, 0]$, then t^* is a singularity of the solution if and only if*

$$\lim_{t \to t^*} \rho(q(t)) = 0$$

We say that a singularity t^* is *due to a collision* if the vector function q has a definite limit when $t \to t^*$. In this case the time t^* is called *collision time*. If q has no limit, or it is unbounded when $t \to t^*$, we call t^* a *pseudocollision* or a noncollision singularity.

Note that if t^* is a collision time, then Theorem 1 tells us that there is at least a mutual distance q_{ij} that vanishes.

In his lectures in Stockholm in 1895 (see [63]), Painlevè proved also the following

THEOREM 2. *(Painlevè) Let $N = 3$ and consider (q, p) to be a solution of the equations defined on $]t^*, 0]$, with $t^* > -\infty$. Then the singularity is due to a collision.*

Therefore noncollision singularities are excluded from the three body problem. Painlevè also stated the following

Conjecture: In the N-body problem with $N \geq 4$ there can exist noncollision singularities.

An important result concerning this problem was given in 1908 by Von Zeipel [76] who proved the following

THEOREM 3. *(Von Zeipel) Let $J : \mathbb{R}^{3N} \to [0, \infty[$ be the function defined by*

$$J(q) = \frac{1}{2}\sum_{i=1}^{N} m_i |q_i|^2 \qquad \textit{(moment of inertia of the system).}$$

Let (q, p) be an analytic solution of the equations (2.2) defined on $]t^*, 0]$ with t^* a singularity and such that $\lim_{t\to t^*} J(q(t))$ is finite. Then the singularity is due to a collision.

Only in recent years the conjecture has been positively solved by Xia [83], who has produced an example of noncollision singularity for $N = 5$. Later Gerver [22] has also shown that for N large enough it is possible to give an example of noncollision singularities in the plane. The existence of noncollision singularities for $N = 4$ is still an open problem.

2.3 The Kepler problem

Let us consider two particles of masses m_1, m_2, whose position is determined by vectors q_1, q_2, moving in the space $\mathbb{R}^3$ under their mutual interactions only, that are described by the potential $U(q) = Gm_1m_2/||q_1 - q_2||$ depending only on the relative position of the particles. This is called the Kepler problem, and has been solved by Newton [62]; in the solution we can find a mathematical proof of the empirical three laws that Johannes Kepler formulated at the beginning of the 17th century, basing his statements on a large amount of observations made by the astronomer Tycho Brahe. We shall speak about the main features of the problem without giving a detailed solution. This will be useful to introduce some relations between the quantities used in the following chapters.

The equations of motion have the form

$$m_1\ddot{q}_1 = \frac{\partial U}{\partial q_1}; \qquad m_2\ddot{q}_2 = \frac{\partial U}{\partial q_2}.$$

One can easily prove the following

Proposition 1. *The relative position vector $\rho = q_1 - q_2$ changes according to the same law as the position vector of a point of mass $\mu = m_1 m_2/(m_1+m_2)$ (*reduced mass*) moving in a central force field with potential $U(\rho) = Gm_1m_2/||\rho||$.*

Let us call

$$\xi = \frac{m_1q_1 + m_2q_2}{m_1+m_2}$$

the position vector of the barycenter of the two points; it follows that

$$q_1 = \xi + \frac{m_2}{m_1+m_2}\rho; \qquad q_2 = \xi - \frac{m_1}{m_1+m_2}\rho;$$

that is the motion of the two particles is completely determined by the relative motion ρ and by the motion of the barycenter ξ. Furthermore from this formulas it follows that in a barycentric coordinate system the trajectories of the two masses are similar planar curves (because $\xi = 0$ and both of them are similar to the trajectory described by ρ) and the two trajectories are on the same plane (*orbital plane*) defined by ρ when the time varies.

By Proposition 1 we have reduced the problem to the single equation

$$\mu\ddot{\rho} = \frac{\partial U}{\partial \rho} \qquad \rho \in \mathbb{R}^3 .$$

If x, y are Cartesian coordinates on the plane of the orbit, then we have

$$K_z = \mu(x\dot{y} - y\dot{x}) = \text{constant}$$

for the third component of the angular momentum and, in polar coordinates $x = r\cos\phi; y = r\sin\phi$, where $r = ||\rho||$, we have $K_z = \mu r^2\dot{\phi}$. We set $\gamma = \rho\wedge\dot{\rho}$ (angular momentum per unit of reduced mass) and $c = r^2\dot{\phi}$ (that is $\pm||\gamma||$). Note that if $c = 0$ then ϕ is constant and the points move along a straight line. Assume that $c \neq 0$, then ϕ is a monotonic function of t and the inverse function $t = t(\phi)$ exists.

The constancy of the angular momentum corresponds to the law of the areas (*Kepler's second law*): we have

$$S(t) = \frac{1}{2}\int_{\phi(0)}^{\phi(t)} r^2\, d\phi = \frac{1}{2}\int_0^t r^2\dot{\phi}\, dt = \frac{ct}{2},$$

where $S(t)$ is the area swept by the position vector as the point moves.

Let us set $\mathcal{U}(r) = \mathcal{U}(||\rho||) = U(\rho)$. The following fact holds:

THEOREM 4. *(Newton) For a fixed value of the area integral c we have*

$$\mu\ddot{r} = \frac{\partial \mathcal{U}_c}{\partial r} \quad \textit{with} \quad \mathcal{U}_c(r) = \mathcal{U}(r) - \frac{c^2}{2r^2} \qquad (r > 0) \ ,$$

that is the problem can be reduced to a unidimensional motion on the half line $\{r > 0\}$ under the action of the *effective potential* $\mathcal{U}_c(r)$.

We define $M = m_1 + m_2$ (*total mass*). The Laplace-Lenz vector

$$\vec{e} = \frac{1}{GM}\dot{\rho} \wedge \gamma - v_\rho \ ,$$

where $v_\rho = \dot{\phi}(-\sin\phi, \cos\phi)$, is a first integral; we can use it to obtain the equation of the trajectories of the solutions of the Kepler problem. We set $e = ||\vec{e}||$ and define the true anomaly f by the equation $e\cos f = \langle \vec{e}, v_\rho \rangle$ (f represents the angle between $\vec{e}$ and ρ as seen from γ). The trajectories of the solutions are described by the equation

$$r = \frac{c^2/(GM)}{1 + e\cos(f)}$$

that tells us that the orbits of the Kepler problem are conics (in polar coordinates) with one focus at the center of attraction (*Kepler's first law*). We are interested in the case of bounded orbits, that is in the case $e < 1$, for which the trajectories are ellipses. If we call a the semimajor axis of the orbit, then we have $c^2/(GM) = a(1-e^2)$.

The elliptic orbits are periodic and the period does not depend on the eccentricity, so that we can define the *mean motion*

$$n = \frac{\sqrt{GM}}{a^{3/2}} \ ,$$

that represents the angular velocity on a circular orbit with semimajor axis a. If we call T the period, then we have $T = 2\pi/n$, so that

$$T^2 = \frac{4\pi^2}{GM}a^3 \qquad (\textit{Kepler's third law}).$$

The complete solution of the Kepler problem is obtained by introducing the *eccentric anomaly u*, defined by

$$x = a(\cos u - e) \ ; \qquad y = a\sqrt{1-e^2}\sin u \ .$$

Using this parametrization we can deduce *Kepler's equation*

$$\ell = n(t - t_0) = u - e\sin u;$$

where t_0 is the time of passage at the pericenter. The variable ℓ defined in the previous equation is called the *mean anomaly*.

2.4 Regularization of collisions in a generalized two body problem

We consider a system of two particles moving under their mutual interactions only. The potential that generates the force is of the form $U(r) = r^{-\alpha}$ where r is the mutual distance of the two particles and $\alpha > 0$ is a real number. This is a generalization of the Kepler problem, discussed in the previous section, that corresponds to $\alpha = 1$. We are interested in collision orbits of this system and in the regularization of the singularities, that is to find whether it is possible to define a continuation of the collision solutions after the singularity.

There are two main points of view in the literature about regularization: the first is the one by K. Sundman [71], that consists in finding a real analytic branch of the solution that can be regarded as a continuation of the solution after the singularity, that in this case can only be a collision.

The second point of view is the one by T. Levi-Civita [42], that consists in changing variables in such a way that the differential equations defining the flow have no singular points: this second method has been reformulated later by Easton [18] using topological tools and we shall present it in this framework. The contents of this section is based on a paper by R. McGehee [45].

2.4.1 The equations of motion

We fix the center of mass at the origin of the reference frame, so that the problem reduces to the one of a single particle that moves in a central force field whose potential is

$$U(x) = |x|^{-\alpha}; \qquad \alpha > 0,$$

where $x \in \mathbb{R}^2$ is the position vector of the particle. Then the equations of motion are

$$\ddot{x} = \nabla U(x) = -\alpha |x|^{-\alpha-2} x \tag{2.3}$$

and the collision singularities correspond to $x = 0$.

2.4.2 Qualitative description of the flow

We can write equations (2.3) as a first order system:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\alpha |x|^{-\alpha-2} x\,; \end{cases}$$

this is a Hamiltonian system, with Hamiltonian $H(x,y) = \frac{1}{2}|y|^2 - |x|^{-\alpha}$.

Let us set $\beta = \alpha/2$ and $\gamma = 1/(\beta+1)$; after the variable change

$$\begin{cases} x = r^{\gamma} e^{i\theta} \\ y = r^{-\beta\gamma}(v + iw) e^{i\theta} \end{cases}$$

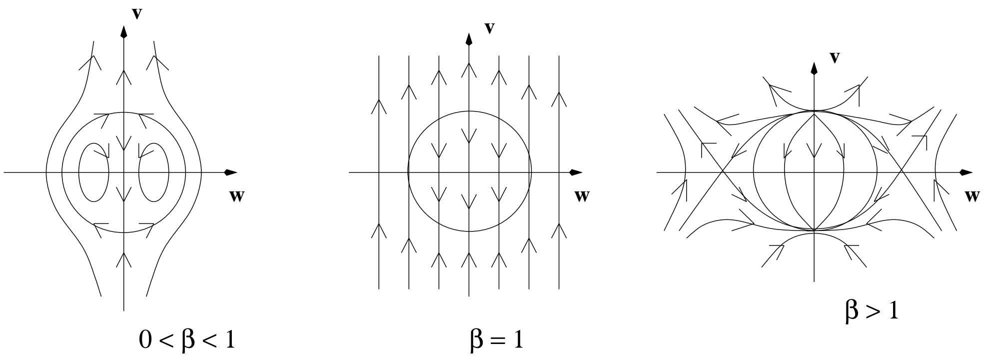

Figure 2.1: The geometry of the flow defined by equations (2.9) is sketched for the three different kinds of behavior: $0<\beta<1, \beta=1$ and $\beta>1$.

and the introduction of the *fictitious time*, defined by $dt = r\,d\tau$, the equations of motion become

$$\begin{cases} r' = (\beta+1)rv \\ \theta' = w \\ w' = (\beta-1)wv \\ v' = w^2+\beta(v^2-2) \end{cases} \tag{2.4}$$

where we use a *prime* to denote the derivative with respect to τ.

The integral of the energy h becomes

$$w^2+v^2-2 = 2hr^{\alpha\gamma} \tag{2.5}$$

and the angular momentum integral c is given by

$$r^{(1-\beta)\gamma}w = c \tag{2.6}$$

We define the constant energy manifold as

$$\mathcal{M}(h) = \{(r,\theta,w,v) \in \mathbb{R}^4 : r \geq 0 \text{ and } (2.5) \text{ holds } \} \tag{2.7}$$

so the manifold of the states corresponding to collisions is

$$\mathcal{N}(h) = \{(r,\theta,w,v) \in \mathcal{M}(h) : r = 0\} \,. \tag{2.8}$$

Note that the last two equations in (2.4) do not depend on r, θ, so that we can study the plane autonomous system defined by

$$\begin{cases} w' = (\beta-1)wv \\ v' = w^2+\beta(v^2-2) \end{cases} \tag{2.9}$$

whose orbit structure is completely determined by the first integral

$$\Lambda(w,v) = |w|^{\alpha}|v^2+w^2-2\,|^{1-\beta} \,.$$

We can distinguish three kinds of different behavior, corresponding to $0 < \beta < 1, \beta = 1$ and $\beta > 1$: we shall describe the geometry of the flow for each of them (see Figure 2.1).

First we notice that the circle $\mathcal{C} = \{(w,v) \in \mathbb{R}^2 : w^2 + v^2 = 2\}$ is invariant and it corresponds to the collision manifold $\mathcal{N}(h)$; the orbits with $h > 0$ project to $\{(w,v) \in \mathbb{R}^2 : w^2 + v^2 \geq 2\}$, the orbits with $h < 0$ to $\{(w,v) \in \mathbb{R}^2 : w^2 + v^2 \leq 2\}$ and the orbits with $h = 0$ to the invariant manifold $\mathcal{C}$.

Let us first consider the case $0 < \beta < 1$. For $h \geq 0$ and $w \neq 0$ (that implies nonzero angular momentum) all the solutions are unbounded in both directions. The orbits with zero angular momentum *either begin or end* into a collision. For $h < 0$ all the solutions are bounded, as you can see from (2.6). There are two critical points $(w,v) = (\pm\sqrt{\alpha}, 0)$ of the reduced flow (2.9), that correspond to stable circular orbits of (2.4).

We notice that the invariant tori in which the phase space is foliated for $\beta < 1$ are generally filled with quasi-periodic orbits, but for $\beta = 1/2$ (the Kepler problem) all the bounded orbits are periodic.

Let us consider the limiting case $\beta = 1$. The invariant manifold $\mathcal{C}$ is composed by critical points. For $h < 0$ each solution *begins and ends* into a collision. Let $h > 0$: if $w \leq \sqrt{2}$ every solution *begins or ends* into a collisions, *even if the angular momentum is different from zero*; if $w > \sqrt{2}$ the solutions are unbounded in both directions. Note that for $h = 0$ also the variable v becomes an integral and for $v = 0$ we have a one parameter family of circular periodic orbits of equations (2.4).

Finally we suppose that $\beta > 1$: the flow is completely different from the previous cases. The circular periodic orbits that correspond to the critical points $(w,v) = (\pm\sqrt{\alpha}, 0)$ are not stable any more, in fact these points are saddles. We have collision orbits and unbounded orbits that are both asymptotic to one of the circular orbits. For $h > 0$ we have unbounded orbits and orbits that *begin and/or end* into a collision. For $h \leq 0$ all the solutions *begin and end* into a collision except for the ones with $h = 0$ and $c = 0$, that only *begin or end* into a collision and become unbounded in the opposite direction.

2.4.3 Branch regularization of the solutions

We present the ideas of Sundman regularization of the collision solutions of (2.3) (*branch regularization*). The singular time t^* can be considered as a *branch point*, if we regard the time as a complex variable. For some values of the positive exponent α we can find a real analytic branch that extends the solution after collisions. We give the following definitions:

Definition 1. *Let $\Phi_1(t), \Phi_2(t)$ be solutions of (2.3). Let us suppose that Φ_1 ends into a singularity at time t^* and that Φ_2 starts in a singularity at the same time. If there exists an analytic multi-valued function that has a* branch point *in t^* and that extends both $\Phi_1(t)$ and $\Phi_2(t)$, then Φ_1 is a* branch extension *of Φ_2 in t^* and Φ_2 is a* branch extension *of Φ_1 in t^*.*

Definition 2. *Let Φ be a solution of equations (2.3) which either begins or ends into a singularity at time t^*. We say that Φ is* branch regularizable *at t^* if it has a unique branch* *extension at t^*. Furthermore we say that equations (2.3) are* branch regularizable *if every solution of it is branch regularizable at every singularity.*

We set

$$Q_0 = \{p/q \in \mathbb{Q} : q \text{ is odd and } 0 < p < q\},$$

with p,q relative prime numbers, and we state the following

THEOREM 5. *Equations (2.3) are* branch regularizable *if and only if* $\gamma \in Q_0$.

A proof of the theorem can be found in [45].

REMARK 2. Indeed in [45] there is the proof of a stronger result: if $\gamma \notin Q_0$, then no singular solution of equations (2.3) is branch regularizable while if $\gamma \in Q_0$, every singular solution of equations (2.3) is branch regularizable.

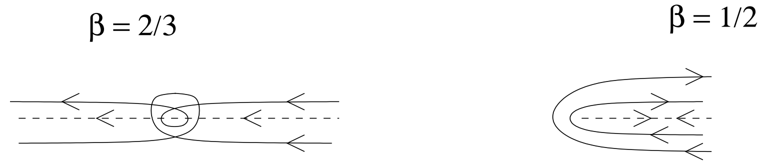

Figure 2.2: The trajectories of the regularized orbits are drawn with hyphened lines: they are represented either by a reflection or by a transmission. We show two examples: $\beta = 2/3$ (left side) and $\beta = 1/2$ (right side), the latter corresponds to the Newtonian case. Trajectories of close orbits are also plotted in the figure.

The trajectory that extends the solution through a collision is a *transmission* (the direction of the velocity vector is preserved at collisions) if p is *odd* and is a *reflection* (the velocity vector reverses direction at collisions) if p is *even*. In Figure 2.2 we show two examples of this different behavior. In the Newtonian case we have $\gamma = 2/3 \in Q_0$, so that the collision solutions are regularizable and the regularized trajectory is a *bounce* with reflection in the collision points.

2.4.4 Block regularization of the solutions

In this section we introduce the geometric notion of regularization due to Easton. We start with the definition of *isolating block*: let $\mathcal{M}$ be a smooth manifold and let

$$\psi : \mathcal{M} \times \mathbb{R} \to \mathcal{M}$$

be a flow on $\mathcal{M}$. A subset $\mathcal{N} \subset \mathcal{M}$ is called *invariant* if $\psi(\mathcal{N}, \mathbb{R}) = \mathcal{N}$.

DEFINITION 3. *A compact invariant set $\mathcal{N} \subset \mathcal{M}$ is called* isolated *if there exists an open set U containing $\mathcal{N}$ such that $x \in \mathcal{M}, \psi(x,\mathbb{R}) \subset U$ implies $x \in \mathcal{N}$. We call the set U an* isolating neighborhood *for $\mathcal{N}$.*

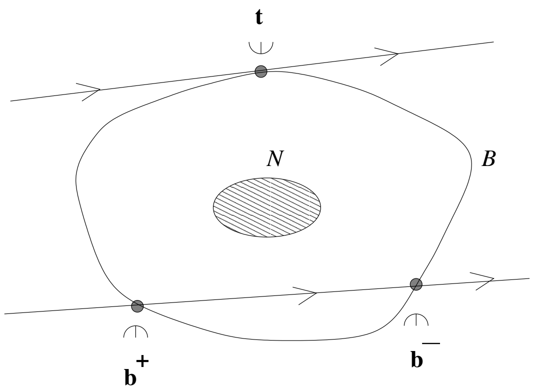

Figure 2.3: $\mathcal{N}$ is an isolated invariant sets, while $\mathcal{B}$ is an isolating block that isolates $\mathcal{N}$.

Let $\mathcal{B}$ be a compact subset of $\mathcal{M}$ with non empty interior and let us assume that $b = \partial\mathcal{B}$ (the boundary of $\mathcal{B}$) is a smooth sub-manifold of $\mathcal{M}$. We define

$$\begin{array}{lcll} b^+ & = & \{x \in b : \psi(x,]-\varepsilon, 0[) \cap \mathcal{B} = \emptyset \text{ for some } \varepsilon > 0\} & (\textit{ingress points of b }) \\ b^- & = & \{x \in b : \psi(x,]0, \varepsilon[) \cap \mathcal{B} = \emptyset \text{ for some } \varepsilon > 0\} & (\textit{egress points of b }) \\ t & = & \{x \in b : \dot\psi(x, 0) \text{ is tangent to } b\} & (\textit{tangency points }) \end{array} \tag{2.10}$$

DEFINITION 4. *If $t = b^+ \cap b^-$ we say that $\mathcal{B}$ is an* isolating block .

DEFINITION 5. *Let $\mathcal{N}$ be an isolated invariant set, and let $\mathcal{B}$ be an isolating block. Then $\mathcal{B}$ is said to* isolate $\mathcal{N}$ *if $int(\mathcal{B})$ (the interior part of $\mathcal{B}$) is an isolating neighborhood for $\mathcal{N}$.*

THEOREM 6. *(Conley and Easton [12]) If $\mathcal{N}$ is an isolated invariant set, then there exists an isolating block which isolates $\mathcal{N}$. If $\mathcal{B}$ is an isolating block, then there exists an isolated invariant set (possibly empty) which is isolated by $\mathcal{B}$.*

We define the subsets of b that are asymptotic to $\mathcal{N}$ as

$$a^+ = \{x \in b^+ : \psi(x, [0, +\infty[) \subset \mathcal{B}\};$$
$$a^- = \{x \in b^- : \psi(x,]-\infty, 0]) \subset \mathcal{B}\}.$$

We notice that, by definition, if $x \in b^+ \setminus a^+$, then there exists a $t > 0$ such that $\psi(x,t) \notin \mathcal{B}$, so that we can define the *time spent in a block* for a point $x \in b^+ \setminus a^+$ as

$$T(x) = \inf\{t > 0 : \psi(x,t) \notin \mathcal{B}\}$$

We also observe that $\psi(x, [0, T(x)]) \in \mathcal{B}$ and that $\psi(x, T(x)) \in b^-$. Now we define the *map across the block* as the application

$$\Psi : b^+ \setminus a^+ \to b^-$$

that sends each $x \in b^+ \setminus a^+$ into $\psi(x, T(x))$. It is also obvious that the image of the points of Ψ cannot belong to a^- because they have not spent all their past time in $\mathcal{B}$.

THEOREM 7. *(Conley and Easton [12]) If $\mathcal{B}$ is an isolating block, then the application $\Psi : b^+ \setminus a^+ \to b^- \setminus a^-$ is a diffeomorphism.*

DEFINITION 6. *An isolating block $\mathcal{B}$ is said to be* trivializable *if the map Ψ extends uniquely to a diffeomorphism $b^+ \to b^-$.*

DEFINITION 7. *Let $\mathcal{B}$ isolate $\mathcal{N}$. Then $\mathcal{N}$ is said to be trivializable if $\mathcal{B}$ is trivializable.*

REMARK 3. The previous definition is well posed because of the following property, that follows from techniques by Conley [11]:

PROPOSITION 2. Suppose that $\mathcal{N}$ is an isolated invariant set and that $\mathcal{B}_1$ and $\mathcal{B}_2$ isolate $\mathcal{N}$. Then $\mathcal{B}_1$ is trivializable if and only if $\mathcal{B}_2$ is trivializable.

Let us consider the central force problem: we take into account the flow ψ on $\mathcal{M}(h)$, defined in (2.7), determined by equations (2.4). The invariant set $\mathcal{N}$ is given by (2.8). In [45] there is the proof of the following

THEOREM 8. *The set $\mathcal{N}$ given by equations (2.8) is a trivializable isolated invariant set for equations (2.4) if and only if $\beta = 1 - n^{-1}$, where n is a positive integer.*

Now we can give Easton's definition of regularization: let $\mathcal{M}$ be a smooth manifold, let $\mathcal{N}$ be a compact subset of $\mathcal{M}$ and let F be a vector field on $\mathcal{M} \setminus \mathcal{N}$. In this case $\mathcal{N}$ is the set of singularities of the vector field F, while in the previous definitions it was an invariant set for the flow (and in particular the flow was defined on $\mathcal{N}$). We consider the flow ϕ on $\mathcal{M} \setminus \mathcal{N}$ given by the vector field F. Let $\mathcal{B}$ be a compact set with not-empty interior part and assume that $b = \partial\mathcal{B}$ is a smooth sub-manifold of $\mathcal{M}$ that does not intersect $\mathcal{N}$. The subsets b^+, b^- are defined as in (2.10).

We define the orbit through x as

$$O(x) = \{\phi(x,t) : \phi(x,t) \text{ is defined } \} \ .$$

DEFINITION 8. *We say that an isolating block $\mathcal{B}$* isolates the singularity set *$\mathcal{N}$ if $\mathcal{N} \subset int(\mathcal{B})$ and if $O(x) \not\subset \mathcal{B}$ for all $x \in \mathcal{B} \setminus \mathcal{N}$.*

The sets a^+, a^- are defined in a similar way; we only have to take into account that in this case the solutions may be not defined for all t:

$$\begin{aligned} a^+ &= \{x \in b^+ : \phi(x,t) \in \mathcal{B} \text{ for all } t \geq 0 \text{ for which } \phi(x,t) \text{ is defined } \} \, ; \\ a^- &= \{x \in b^- : \phi(x,t) \in \mathcal{B} \text{ for all } t \leq 0 \text{ for which } \phi(x,t) \text{ is defined } \} \ . \end{aligned}$$

We define a map $\Phi : b^+ \setminus a^+ \to b^- \setminus a^-$ in the same way as before and we give the following

DEFINITION 9. *The singularity set $\mathcal{N}$ is said* block regularizable *if there exists a trivializable block $\mathcal{B}$ which isolates $\mathcal{N}$.*

Theorem 8 tells us that

THEOREM 9. *The singularity set $\mathcal{N}$ for system (2.4) is block regularizable if and only if $\beta = 1 - n^{-1}$, where n is a positive integer.*

Chapter 3

Perturbation theory and the averaging principle

3.1 Integrable systems and action-angle variables

We say that a system of Ordinary Differential Equations (ODEs) is *integrable* if its solutions can be expressed by analytic formulas up to inversions (by the implicit function theorem) or quadratures; we call the system *non-integrable* if this is not possible.

It is easy to show that if we consider a system of $2n$ differential equations (where n is an integer) and we have $2n$ first integrals, then we can integrate the equations and solve the problem. If the system is canonical, in several cases we only need to know n first integrals satisfying some properties to integrate the system. We have in fact the following:

THEOREM 10. *(Liouville-Arnold) Let $\mathcal{M}$ be a symplectic manifold $2n$-dimensional and let $F_i : \mathcal{M} \to \mathbb{R}, i = 1 \dots n$ be functions, defined on $\mathcal{M}$, that are* in involution*, that is the Poisson brackets $\{F_i, F_j\}$ vanish for each $0 \leq i < j \leq n$. We consider the level set of the F_i:*

$$\mathcal{M}_f = \{x : F_i(x) = f_i, i = 1 \dots, n\}.$$

Let us assume that the F_i are independent on $\mathcal{M}_f$ (the 1-forms dF_i are linearly independent in each point of $\mathcal{M}_f$). Then we can conclude that $\mathcal{M}_f$ is a smooth manifold, invariant with respect to the phase flow with Hamilton function $H = F_1$.

If $\mathcal{M}_f$ is a compact connected manifold, it is diffeomorphic to the n-dimensional torus

$$\mathbb{T}^n = \{(\phi_1, \dots, \phi_n) \; mod \; 2\pi\},$$

and

1. *the phase flow with Hamilton function H determines on $\mathcal{M}_f$ a quasi-periodic motion, that is, using angular variables $\phi = (\phi_1, \dots, \phi_n)$, we have*

$$\dot{\phi} = \omega; \qquad \omega = \omega(f);$$

2. *the canonical equations with Hamilton function H can be integrated.*

If we are in the hypotheses of Theorem 10 we can select symplectic coordinates (I, ϕ) such that the first integrals F depend only on the coordinates I (*action* variables), while ϕ (*angle* variables) are angular coordinates on the torus $\mathbb{T}^n$.

3.2 Perturbation methods

Most of the problems in Dynamics, included the three body problem (see [64]), are not integrable, anyway several of them can be represented by differential equations that are a small perturbation of an integrable problem. Let $\mathcal{M}$ be a smooth manifold, let $F, G : \mathcal{M} \longrightarrow T\mathcal{M}$ be smooth maps from the manifold $\mathcal{M}$ to its tangent bundle $T\mathcal{M}$ and let ε be a natural small parameter; we can consider the system

$$\dot{X} = F(X) + \varepsilon G(X) \qquad (\textit{perturbed problem}), \tag{3.1}$$

obtained by perturbing the integrable system

$$\dot{X} = F(X) \qquad (\textit{unperturbed problem}).$$

Even if the system (3.1) is non integrable, we can study it in a qualitative sense, that is we can study some properties of the solutions or we can try to approximate them. Perturbation theory consists of the methods that allow to approximate the solutions of a perturbed problem, like (3.1), starting from the solutions of the unperturbed one.

3.3 The averaging principle

The averaging principle is a powerful tool to study the qualitative behavior of the solutions of Ordinary Differential Equations. It consists in solving averaged equations, obtained by an integral average of the original equations over some angular variables; then we consider the solutions of the averaged equations as representative of the solutions of the original equations for a long time span (of the order of $1/\varepsilon$). A review of the classical results on averaging methods in perturbation theory can be found in [2].

Let us assume that $\mathcal{M} = \Omega \times \mathbb{T}^n$ where Ω is an open set in $\mathbb{R}^k$ (with k integer). We consider the *unperturbed equations*

$$\begin{cases} \dot{\phi} = \omega(I) \\ \dot{I} = 0 \end{cases}$$

and the corresponding *perturbed equations*

$$\begin{cases} \dot{\phi} = \omega(I) + \varepsilon f(I, \phi, \varepsilon) \\ \dot{I} = \qquad\quad\ \varepsilon g(I, \phi, \varepsilon) \end{cases} \tag{3.2}$$

where $f(I, \phi, \varepsilon), g(I, \phi, \varepsilon) \in C^1(\Omega \times \mathbb{T}^n \times [0, 1])$ are 2π-periodic function in the variable $\phi \in \mathbb{T}^n$.

We define as *averaged equations*

$$\dot{J} = \varepsilon G(J)$$

where

$$G(J) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \dots \int_0^{2\pi} g(J, \phi, 0)\, d\phi_1 \dots d\phi_n$$

is the *spatial average* over the torus $\mathbb{T}^n$.

Let us consider a time interval $[0, T]$, with $1 \ll T \ll 1/\varepsilon$, that is T is big with respect to 1, and small if compared to $1/\varepsilon$. The increment in the I variable after time T has elapsed is approximated by

$$\Delta I = \varepsilon T \left[\frac{1}{T} \int_0^T g(I, \phi(t), 0)\, dt \right] + o(\varepsilon T) \tag{3.3}$$

where $I(t), \phi(t)$ are the solutions of the unperturbed problem, that is $I(t) = I(0)$ is constant (we have called it I for brevity) and $\phi(t) = \phi(0) + \omega(I)t$.

We define the *time average* of the function g as

$$g^*(\phi(0)) = \lim_{T \to +\infty} \frac{1}{T} \int_0^T g(I, \phi(t), 0)\, dt$$

(it obviously depends on the initial phase $\phi(0)$) and we notice that the expression between the square parentheses in (3.3) is representative of the time average of the function g as the time T is large. If we introduce the *slow time* $\tau = \varepsilon t$, we can write

$$\frac{\Delta I}{\Delta \tau} \approx g^*(\phi(0))$$

and, if we consider the limit for $\varepsilon T \to 0$ and we use a prime for the derivation with respect to the slow time, we obtain

$$I' = g^*(\phi(0)). \tag{3.4}$$

From relation (3.4) we can see that the passage to the averaged equations corresponds to the substitution of the time average with the spatial average.

3.3.1 Averaging in 1-frequency systems

We assume that the phase space D is the direct product of an open set $\Omega \subseteq \mathbb{R}^n$ and the circle S^1; then we can take into account the average of the perturbed equations (3.2) over the angular variable $\phi \in [0, 2\pi[$. In this case the averaged equations are

$$\dot{J} = \varepsilon G(J); \qquad G(J) = \frac{1}{2\pi} \int_0^{2\pi} g(J, \phi, 0)\, d\phi\,. \tag{3.5}$$

Theorem 11. Let $I_0 \in \Omega \subseteq \mathbb{R}^n$ and let us call $I(t)$ the solution of equations (3.2) and $J(t)$ the solution of the averaged equations (3.5) starting from the same initial conditions $I(0) = J(0) = I_0$. If $\varepsilon > 0$ is small enough, if $\omega(I) \neq 0$ for all $I \in \Omega$ and if $J(t)$ remains in a compact domain $K \subset \Omega$ for $t \in [0, T/\varepsilon]$, then there exists a constant C, *independent from* ε, such that

$$|I(t) - J(t)| \leq \varepsilon C$$

for all $t \in [0, T/\varepsilon]$.

In other words, provided that ε is small enough, the difference between the solution of the perturbed equations and the solution of the averaged equations, starting from the same initial conditions, will remain small during a time span T/ε.

Proof. This proof is based on the one given in [1]. We divide it into 5 steps:

Step 1 We shall see that if ε is small enough, it is possible to define a variable change

$$\Psi_\varepsilon : (I, \phi) \longrightarrow (P, \phi)$$

where

$$P = P(I, \phi, \varepsilon) = I + \varepsilon h(I, \phi) \tag{3.6}$$

and $h(I, \phi)$ is 2π-periodic function in the variable ϕ. We want to select $h(I, \phi)$ in order to eliminate the dependence on ϕ from the differential equation for P, up to the first order in ε included. Let us differentiate equation (3.6) with respect to time: we obtain

$$\begin{aligned}
\dot{P} &= \dot{I} + \varepsilon \left[\frac{\partial h}{\partial I}(I, \phi)\dot{I} + \frac{\partial h}{\partial \phi}(I, \phi)\dot{\phi} \right] = \\
&= \varepsilon g(I, \phi, \varepsilon) \left[1 + \varepsilon \frac{\partial h}{\partial I}(I, \phi) \right] + \varepsilon \left[\omega(I) + \varepsilon f(I, \phi, \varepsilon) \right] \frac{\partial h}{\partial \phi}(I, \phi) = \qquad (3.7) \\
&= \varepsilon \left[g(I, \phi, \varepsilon) + \omega(I) \frac{\partial h}{\partial \phi}(I, \phi) \right] + \varepsilon^2 r(I, \phi, \varepsilon)
\end{aligned}$$

where

$$r(I, \phi, \varepsilon) = \left[g(I, \phi, \varepsilon) \frac{\partial h}{\partial I}(I, \phi) + f(I, \phi, \varepsilon) \frac{\partial h}{\partial \phi}(I, \phi) \right] .$$

We search for a function $h(I, \phi)$ such that

$$\left[g(I, \phi, \varepsilon) + \omega(I) \frac{\partial h}{\partial \phi}(I, \phi) \right] = 0 \ :$$

let us set

$$h(I, \phi) = -\frac{1}{\omega(I)} \int_0^\phi \tilde{g}(I, \psi)\, d\psi \tag{3.8}$$

with

$$\tilde{g}(I, \phi) = g(I, \phi, 0) - G(I).$$

so that $h(I, \phi)$ is 2π-periodic in ϕ ($\tilde{g}$ has zero average), and the following relation holds:

$$g(I, \phi, \varepsilon) + \omega \frac{\partial h(I, \phi)}{\partial \phi} = g(I, \phi, \varepsilon) - g(I, \phi, 0) + G(I) \, . \tag{3.9}$$

Using (3.9) we can write equation (3.7) as follows

$$\dot{P} = \varepsilon\,[G(I) + g(I,\phi,\varepsilon) - g(I,\phi,0)] + \varepsilon^2\, r(I,\phi,\varepsilon)\ . \tag{3.10}$$

Let $K \subset \Omega$ be a *compact, convex* set containing I_0 and such that $J(t) \in K$ for each $t \in [0, T/\varepsilon]$; for each function $u = u(I,\phi,\varepsilon) \in C^1(\Omega \times S^1 \times [0,1])$ we set

$$\begin{aligned} |u|_0 &= \max_{(I,\phi,\varepsilon)\in K\times S^1\times[0,1]} |u(I,\phi,\varepsilon)|\ ; \\ |u|_1 &= |u|_0 + \max_{(I,\phi,\varepsilon)\in K\times S^1\times[0,1]} \left| \frac{\partial}{\partial(I,\phi,\varepsilon)} u(I,\phi,\varepsilon) \right|\ . \end{aligned}$$

REMARK 4. When we consider functions $v = v(I,\phi)$, not depending on ε we shall use the notation $|v|_0 := |\tilde{v}|_0$ and $|v|_1 := |\tilde{v}|_1$ where $\tilde{v}(I,\phi,\varepsilon) := v(I,\phi)$.

We select a constant c_1 such that

$$|f|_1 \le c_1; \qquad |g|_1 \le c_1; \qquad |\omega^{-1}|_1 \le c_1;$$

$(\omega(I,\phi) = \omega(I))$.

Step 2 Let us show that the application $\Psi_\varepsilon : (I,\phi) \longrightarrow (P,\phi)$ is a diffeomorphism for ε small enough. From Definition 3.8 we have $h \in C^1(\Omega \times S^1)$ and $|\varepsilon h|_1 < 1$ for ε small enough. First we prove that Ψ_ε is one-to-one: if $\Psi_\varepsilon(I_1,\phi_1) = \Psi_\varepsilon(I_2,\phi_2)$ then $\phi_1 = \phi_2 := \phi$ and

$$I_1 + \varepsilon h(I_1,\phi) = I_2 + \varepsilon h(I_2,\phi).$$

This means

$$\varepsilon h(I_1,\phi) - \varepsilon h(I_2,\phi) = I_2 - I_1\ ,$$

that is in contradiction with the relation $|\varepsilon h|_1 < 1$, that gives an upper bound to the Lipschitz constant of the function εh (note that the convexity of the domain K is needed at this step to use Lagrange's theorem and obtain a Lipschitz constant smaller than 1).

From $|\varepsilon h|_1 < 1$ also follows that Ψ_ε is a local diffeomorphism, in fact

$$\left|\frac{\partial P(I,\phi,\varepsilon)}{\partial I}\right| = \left|\frac{\partial}{\partial I}(I + \varepsilon h(I,\phi))\right| = \left|1 + \frac{\partial}{\partial I}(\varepsilon h(I,\phi))\right| > \left|1 - \left|\frac{\partial}{\partial I}(\varepsilon h(I,\phi))\right|\right| > |1 - |\varepsilon h|_1| > 0$$

for each $(I,\phi,\varepsilon) \in K \times S^1 \times [0,\bar{\varepsilon}]$ with $\bar{\varepsilon} > 0$ small enough, and we are in the hypotheses of the *local inversion theorem*; we can conclude that Ψ_ε is a *diffeomorphism* for ε small enough.

Step 3 Writing the differential equation (3.10) in the (P,ϕ) variables, we obtain

$$\dot{P} = \varepsilon\,[G(P) + R(P,\phi,\varepsilon)]\ . \tag{3.11}$$

We shall show that $R(P,\phi,\varepsilon)$ is an infinitesimal of the first order in ε, so that the differential equation for P, up to terms of the second order in ε, is given by

$$\dot{P} = \varepsilon\, G(P)\ ;$$

we shall do the computations using the variables (I,ϕ). As

$$G(P) = G(I+\varepsilon h(I,\phi)) = G(I) + \varepsilon h(I,\phi)\frac{\partial}{\partial I}G(\xi(P,I))$$

for $\xi(P,I)$ on the line joining I and P, we have

$$\begin{aligned}
&R(P(I,\phi,\varepsilon),\phi,\varepsilon) = -\varepsilon h(I,\phi)\frac{\partial}{\partial I}G(\xi(P,I)) + g(I,\phi,\varepsilon) - g(I,\phi,0) + \varepsilon r(I,\phi,\varepsilon) = \\
&= -\varepsilon h(I,\phi)\frac{\partial}{\partial I}G(\xi(P,I)) + g(I,\phi,\varepsilon) - g(I,\phi,0) + \varepsilon\left[g(I,\phi,\varepsilon)\frac{\partial h}{\partial I}(I,\phi) + f(I,\phi,\varepsilon)\frac{\partial h}{\partial \phi}(I,\phi)\right] .
\end{aligned}$$

We use the following bounds for each $(I,\phi,\varepsilon) \in K \times S^1 \times [0,1]$:

$$\begin{aligned}
&|h(I,\phi)| \le |h|_0 \le 2\pi\left|\omega^{-1}\right|_0|\tilde{g}|_0 \le 4\pi\left|\omega^{-1}\right|_0|g|_0 \,; \\
&\left|\frac{\partial G}{\partial I}(\xi(P,I))\right| \le |g|_1 \,; \\
&|g(I,\phi,\varepsilon) - g(I,\phi,0)| \le \varepsilon|g|_1 \,; \\
&|r(I,\phi,\varepsilon)| \le |g|_0\left|\frac{\partial h}{\partial I}(I,\phi)\right| + |f|_0\left|\frac{\partial h}{\partial \phi}(I,\phi)\right| \,; \\
&\left|\frac{\partial h}{\partial I}(I,\phi)\right| \le 4\pi\left\{|\omega|_1\left|\omega^{-2}\right|_0|g|_0 + \left|\omega^{-1}\right|_0|g|_1\right\} \,; \\
&\left|\frac{\partial h}{\partial \phi}(I,\phi)\right| \le 2\pi\left|\omega^{-1}\right|_0|g|_1 \,;
\end{aligned}$$

and we conclude that the differential equations (3.11) differ from the averaged equations (3.5) only for terms of the second order in ε; in particular there exists a constant c_2, independent on I,ϕ,ε and such that

$$|R(P(I,\phi,\varepsilon),\phi,\varepsilon)| \le \varepsilon c_2$$

for each $(I,\phi,\varepsilon) \in K \times S^1 \times [0,1]$.

Step 4 Let us consider the *slow time* $\tau = \varepsilon t$; if we use a prime for the derivative with respect to τ we obtain that P and J are the solutions of the following differential equations:

$$P' = G(P) + R(P,\phi,\varepsilon)\,; \tag{3.12}$$

$$J' = G(J)\,. \tag{3.13}$$

We estimate the difference between the solution of the averaged equations (3.13) and the function $P = P(t)$, that is solution of the equations (3.12). Let us set $Z(\tau) = P(t(\tau)) - J(t(\tau))$, then, assuming that I,P,J remain in the domain K, we have

$$\left|Z'(\tau)\right| \le a\,|Z(\tau)| + b \tag{3.14}$$

where $a = |G|_1$ and $b = \varepsilon c_2$.

Let us suppose that $|Z(0)| = c$; then by Gronwall's lemma we obtain

$$|Z(\tau)| \leq (c + b\tau)e^{a\tau}, \tag{3.15}$$

because the right hand side of (3.15) is the solution of the Cauchy problem

$$Z' = aZ + b; \qquad Z(0) = c\ .$$

We observe that

$$|c| = |Z(0)| = |P(0) - J(0)| = |I(0) + \varepsilon h(I(0), \phi(0)) - J(0)| = \varepsilon |h(I(0), \phi(0))| < \varepsilon |h|_0\ .$$

Step 5 If we set $c_3 = |h|_0$ we obtain $|P(I, \phi, \varepsilon) - I| \leq \varepsilon c_3$. On the other hand from (3.14) we obtain

$$|P(t) - J(t)| = |Z(\tau)| \leq \varepsilon c_4$$

with $c_4 = (c_3 + c_2 T)e^{aT}$ for all t such that $0 \leq \varepsilon t \leq T$; we conclude that

$$|I(t) - J(t)| \leq |I(t) - P(t)| + |P(t) - J(t)| \leq \varepsilon(c_3 + c_4) \qquad \forall t \in [0, T/\varepsilon]\ .$$

□

3.3.2 Multi-frequency systems and resonances

Let us consider a system with more than one frequency, where the frequencies $\omega = \omega(I)$ depend on the slow variables $I \in B$. The main difficulty when dealing with such systems is the possible presence of resonances between the frequencies:

DEFINITION 10. *Let us consider a system like (3.2); we say that the frequency vector* $\omega = (\omega_1, \ldots, \omega_n)$ *is* resonant *if there exists a vector* $k = (k_1, \ldots, k_n)$ *with integer coefficients such that* $\langle k, \omega \rangle = 0$. *The vector* k *is called the* resonance number.

DEFINITION 11. *A point* $I \in B$ *is called* resonant *if the vector* $\omega(I)$ *is resonant. We define, for each resonant number* k*, the* resonant hypersurface

$$\Gamma_k = \{I \in B : \langle k, \omega(I) \rangle = 0\}\ .$$

DEFINITION 12. *We say that the domain is* non-resonant *if for each* $I \in B$ *the condition*

$$|\langle k, \omega(I) \rangle| > c^{-1}|k|^{-\nu} \qquad (\text{strong incommensurability}) \tag{3.16}$$

holds for some constants c, ν *and for all the vectors* $k \neq 0$ *with integer components such that the harmonic with phase* $\langle k, \phi \rangle$ *appears in the Fourier expansion of the function* f *in equations (3.2).*

If the domain B is non-resonant, then we can apply the averaging principle and we obtain an accuracy of the order ε over times of the order $1/\varepsilon$ for the results.

We can also replace the strong incommensurability condition (3.16) with the weaker condition $\langle k, \omega(I) \rangle \neq 0$, but we have to take into account a possible loss in accuracy.

REMARK 5. Note that in most cases in a multi-frequency system there are no non-resonant domains because the incommensurability conditions is generally violated on an everywhere dense set.

Then we have to face the problem of crossing the resonant surfaces. We notice that some resonances are more important than others, in particular the effects of the resonances decrease with their orders (we say that the number $|k| = \sum_{i=1}^{n} |k_i|$ is the order of the resonance defined by $k \in \mathbb{Z}^n$).

We observe that in a multi-frequency system there is the possibility of a *capture in resonance* for the solutions; in such cases the averaged system is not representative of the behavior of the projection of the basis B of the solutions of the full equations. Anyway the capture in resonance can happen only for an exceptional set of initial conditions.

A case that has been intensively studied is the case of two frequencies only. Let us consider the system

$$\begin{cases} \dot{I} = \varepsilon f(I, \phi, \varepsilon) \\ \dot{\phi}_1 = \omega_1(I) + \varepsilon g_1(I, \phi, \varepsilon) \\ \dot{\phi}_2 = \omega_2(I) + \varepsilon g_2(I, \phi, \varepsilon) \end{cases} \tag{3.17}$$

and let us assume that the functions at the right hand sides of (3.17) are analytic. The following result holds:

THEOREM 12. *Suppose that*

$$\left(\omega_1 \frac{\partial \omega_2}{\partial I} - \omega_2 \frac{\partial \omega_1}{\partial I}\right) f > c_1^{-1} > 0$$

holds true. Then the difference between the slow motion in the perturbed system, $I(t)$, and in the averaged system, $J(t)$, remains small over time $1/\varepsilon$; if $I(0) = J(0)$, then

$$|I(t) - J(t)| \leq c_2 \sqrt{\varepsilon}, \qquad 0 \leq t \leq 1/\varepsilon\,.$$

The case with more that two frequencies has been studied less because the arrangement of the resonant surfaces in the phase space is more complicate. In the case of two frequencies the main effect is a passage through a single resonance, while in the case of more than two frequencies we have to take into account the tangency to the resonant surfaces and the addition of the effects of several resonances (see [2] for a more detailed explanation).

However there exist some theorems that justify in a statistical sense the applicability of the averaging methods also in the multi-frequency case: we give the statement of one of them. Let us assume that for almost all the values of the integrals of the unperturbed problem, the unperturbed motion on the common level set is ergodic; then we have

THEOREM 13. (Anosov) For each $\rho > 0$, the measure of the set of initial conditions (in a compact subset of the phase space) for which the error made in describing the exact motion by averaging exceeds ρ, i.e.,

$$mes\{(I_0, \phi_0) : \max_{0 \leq t \leq 1/\varepsilon} |I(t) - J(y)| > \rho \text{ if } I(0) = J(0) = I_0\},$$

tends to zero as $\varepsilon \to 0$.

3.3.3 Averaging in Hamiltonian systems

We consider the study of small Hamiltonian perturbations of a completely integrable Hamiltonian system, that was called by Poincarè [64] *the fundamental problem of dynamics*. Let us set

$$H(I,\phi,\varepsilon) = H_0(I) + \varepsilon H_1(I,\phi,\varepsilon)$$

where $H_1(I,\phi,\varepsilon)$ is a 2π-periodic function in the variable ϕ. The Hamiltonian $H_0(I)$ of the unperturbed system depends only on the action variables I so that the equations of the unperturbed motion are

$$\begin{cases} \dot{I} &= 0 \\ \dot{\phi} &= \dfrac{\partial H_0}{\partial I} \end{cases}$$

and the equations of the perturbed motion are

$$\begin{cases} \dot{I} = -\varepsilon \dfrac{\partial H_1}{\partial \phi} \\ \dot{\phi} = \dfrac{\partial H_0}{\partial I} + \varepsilon \dfrac{\partial H_1}{\partial I} \end{cases} .$$

PROPOSITION 3. *In a Hamiltonian system with n degrees of freedom and n frequencies, evolution of slow variables does not occur, in the sense that the averaged system has the form* $\dot{J} = 0$.

Proof. The integral $\int_0^{2\pi} [-(\partial H_1/\partial\phi)]\, d\phi$ yields zero, because it is the increment of the periodic function H_1 over its period. □

REMARK 6. The averaged equation is equal to the differential equation for I in the unperturbed system.

3.3.4 Averaging in Hamiltonian systems in the degenerate case

Often we encounter *properly degenerate* problems. in which the unperturbed Hamiltonian depends only on a part of the action variables and some of the unperturbed frequencies vanish identically:

$$H = H_0(I_1,\dots,I_r) + \varepsilon H_1(I,\phi,\varepsilon) \qquad r < n\ . \tag{3.18}$$

As the angles ϕ_j with $j > r$ are slow variables the averaging method can represent the evolution of the system if it is done only on the fast phases ϕ_i with $i \le r$.

We have the following

PROPOSITION 4. *In a Hamiltonian system with n degrees of freedom, with Hamiltonian of the form (3.18), we have $r < n$ fast frequencies and the variables canonically conjugate to the fast phases are first integrals of the averaged system.*

We say that the degeneracy of the system is *simple* if the number of the fast phases is smaller by only one unit than the number of the degrees of freedom of the system: in this case the averaged system is integrable.

As the Hamiltonian of the Solar System planets is degenerate, we can deduce from Theorem 4 the following

COROLLARY 1. (Lagrange's theorem) *The semimajor axes of the Solar system planets do not present a secular evolution, up to terms of the first order in the perturbing masses (the ratio of the masses of the planets with respect to the mass of the Sun).*

3.4 Perturbative formulation of the restricted three body problem

In 1772, while studying the three body problem Sun-Earth-Moon, Leonard Euler thought to simplify the problem by assuming that the two more massive bodies had a circular motion around their common center of mass and that the Moon moved under the influence of the two bodies without perturbing their motion: this is called *circular restricted three body problem*. Even with these simplifications, the three body problem turns out to be not integrable, nevertheless the restricted circular problem has been used successfully for qualitative investigations of the motion of the Solar System bodies; the approximation that is made disregarding the influence of the third body makes even more sense if this body is an asteroid.

Let us consider a restricted 3-body problem Sun-planet-asteroid (we allow also elliptic orbits for the Sun and the planet): the gravitational attraction of the asteroid on the other two bodies is neglected and the equations to solve, in an inertial reference frame $Oy_1y_2y_3$, are

$$\begin{cases} \ddot{y} = k^2 \left\{ \dfrac{y_\odot - y}{|y_\odot - y|^3} + \mu \dfrac{y' - y}{|y' - y|^3} \right\} \\ \ddot{y}' = k^2 \mu \dfrac{y_\odot - y'}{|y_\odot - y'|^3} \\ \ddot{y}_\odot = k^2 \mu \dfrac{y' - y_\odot}{|y' - y_\odot|^3} \end{cases}$$

where $y, y', y_\odot$ are the positions of the asteroid, the planet and the Sun respectively, k^2 is Gauss's constant and $\mu = m'/m_\odot$ that is the ratio between the mass of the planet and the mass of the Sun.

The last two equations are independent from the first and the motion of the Sun and of the planet is completely determined as solution of a 2-body problem. Therefore the problem consists only in determining the motion of the asteroid around the Sun.

Perturbation theory is used to compute an approximation of the solution of this non-integrable problem.

In a heliocentric reference frame $O'x_1x_2x_3$, oriented like $Oy_1y_2y_3$, the equations of motion for the asteroid are

$$\ddot{x} = k^2\Big\{-\frac{x}{|x|^3}+\mu\frac{x'-x}{|x'-x|^3}-\mu\frac{x'}{|x'|^3}\Big\} \tag{3.19}$$

where $x = y - y_\odot$ and $x' = y - y'$.

The right hand side of equations (3.19) is the gradient of a *potential* $U = U_0 + \mu U_1$ where

$$U_0 = \frac{k^2}{|x|} \qquad U_1 = k^2\Big[\frac{1}{|x-x'|} - \frac{< x, x' >}{|x'|^3}\Big]$$

($<,>$ is the usual Euclidean scalar product) so that equations (3.19) can be written in the form

$$\ddot{x} = \nabla_x(U_0 + \mu U_1)\ .$$

We observe that the mass of Jupiter, the most massive planet in the Solar System, is only about $1/1000$ of the mass of the Sun, so that we can regard μ as a small parameter: hence we are dealing with a perturbed problem of the integrable two body motion of the asteroid.

3.5 The secular evolution of Main Belt Asteroids

Let us consider a Solar System model with the Sun, $N-2$ planets and an asteroid and let the mass of the asteroid be negligible, so that we have a *restricted* problem. Let us suppose in addition that the mass of the planets is small if compared to the mass of the Sun, then we have $N-2$ small perturbative parameters $\mu_i, i = 1\dots N-2$.

We assume that the motion of the planets is completely determined and that there are no collisions between them or with the Sun.

We can apply the averaging principle in order to study the qualitative behavior of the orbits of Main Belt Asteroids (MBAs) assuming that *no mean motion resonances with low order occur* occurs between the asteroid and the planets in the model. This means that there exists $\varepsilon > 0$ not too small and a positive integer M not too large such that for each pair $(a(t), a_i(t))$, composed by the semimajor axes of the osculating orbits of the asteroid and the planet i $(i = 1\dots N-2)$ we have

$$\Big|\ p\,[a(t)]^{3/2} - q\,[a_i(t)]^{3/2}\ \Big| > \varepsilon$$

for each pair of positive integers $p, q \leq M$ and for each t in the considered time span.

Remark 7. As our purpose is not a study of the structure of the mean motion resonances we shall not give further details or any estimates on the size of ε and M.

We can study the motion of the asteroid using the heliocentric Delaunay variables (L,G,Z,ℓ,g,z), defined by

$$\begin{cases} L = k\sqrt{a} \\ G = k\sqrt{a(1-e^2)} \\ Z = k\sqrt{a(1-e^2)}\cos I \end{cases} \qquad \begin{cases} \ell = n(t-t_0) \\ g = \omega \\ z = \Omega \end{cases}$$

where $\{a,e,I,\omega,\Omega,\ell\}$ is the set of the Keplerian elements, k is Gauss's constant, n is the mean motion and t_0 is the time of passage at perihelion; the same variables with a suffix $(L_i,G_i,Z_i,\ell_i,g_i,z_i)$ $i = 1\dots N-2$ can be used to describe the motion of the planets.

Delaunay's variables, like the Keplerian elements, describe the evolution of the osculating orbit of the asteroid, that is of the trajectory that the asteroid would describe in a heliocentric reference frame, given its position and velocity at a time t, if only the Sun were present. For negative values of the Keplerian energy of the asteroid the osculating orbits are ellipses; we shall consider only such cases.

The Hamiltonian can be written as

$$H = -\frac{k^2}{2L^2} - R \tag{3.20}$$

where $-k^2/(2L^2)$ is the *unperturbed term*, describing the two body motion of the asteroid around the Sun, and R is the *perturbing function* defined by

$$R = \sum_{i=1}^{N-2} \mu_i R_i; \quad R_i = k^2\Big[\frac{1}{|x-x_i|} - \frac{< x,x_i >}{|x_i|^3}\Big]; \quad i = 1\dots N-2 \tag{3.21}$$

in which $\langle\,,\rangle$ is the Euclidean scalar product and x and x_i are the position vectors of the asteroid and of all the planets in a heliocentric reference frame.

Note that each R_i is the sum of a *direct term* $k^2/|x-x_i|$, due to the direct interaction between the planet i and the asteroid, and an *indirect term* $-k^2 < x,x_i > /|x_i|^3$, representing the effects on the motion of the asteroid caused by the interaction between the Sun and the planet i.

If we set $\mathfrak{E}_{\mathcal{D}} = (L,G,Z,\ell,g,z)$ we can write Hamilton's equations as

$$\dot{\mathfrak{E}}_{\mathcal{D}} = \mathfrak{J}\,(\nabla_{\mathfrak{E}_{\mathcal{D}}} H)^t \tag{3.22}$$

where $\mathfrak{J}$ is the 6×6 matrix

$$\begin{bmatrix} O_3 & -I_3 \\ I_3 & O_3 \end{bmatrix}$$

composed by 3×3 zero and identity matrices, and

$$(\nabla_{\mathfrak{E}_{\mathcal{D}}} H)^t = \left(\frac{\partial H}{\partial \mathfrak{E}_{\mathcal{D}}}\right)^t$$

is the transposed vector of the partial derivatives of the Hamiltonian H with respect to $\mathfrak{E}_{\mathcal{D}}$.

3.5.1 The averaged equations

In this case we can consider averaged equations obtained by the integral average of the right hand side of (3.22) over the mean anomalies $\ell, \ell_1, .., \ell_{N-2}$ of the asteroid and the planets.

In the expression of the perturbing function (3.21) the effect of each planet is independently taken into account: each R_i is a function of the coordinates and the masses of the asteroid and one planet only. We shall study the case of only one perturbing planet and we shall use a prime for the quantities related to this planet: the perturbation of all the planets, up to the first order in the perturbing masses μ_i, will be obtained as the sum of the contribution of each planet.

Let us consider a three body restricted problem Sun-planet-asteroid; If we take into account the reduced set of Delaunay's variables $\mathcal{E}_{\mathcal{D}} = (G, Z, g, z)$, the averaged equations of motion for the asteroid can be written in the following form:

$$\dot{\widetilde{\mathcal{E}}}_{\mathcal{D}} = -\mathcal{J}\, \overline{\nabla_{\mathcal{E}_{\mathcal{D}}} R}^{\,t} \tag{3.23}$$

where $\widetilde{\mathcal{E}}_{\mathcal{D}} = (\widetilde{G}, \widetilde{Z}, \tilde{g}, \tilde{z})$ are averaged Delaunay variables, $\mathcal{J}$ is the 4×4 matrix

$$\mathcal{J} = \begin{bmatrix} O_2 & -I_2 \\ I_2 & O_2 \end{bmatrix}$$

composed by 2×2 zero and identity matrices, and $\overline{\nabla_{\mathcal{E}_{\mathcal{D}}} R}^{\,t}$ is the transposed vector of the integral average over (ℓ, ℓ') of the partial derivatives of the perturbing function R with respect to $\mathcal{E}_{\mathcal{D}}$

$$\overline{\nabla_{\mathcal{E}_{\mathcal{D}}} R} = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \nabla_{\mathcal{E}_{\mathcal{D}}} R \, d\ell \, d\ell' \; ; \qquad \nabla_{\mathcal{E}_{\mathcal{D}}} R = \frac{\partial R}{\partial \mathcal{E}_{\mathcal{D}}} \, .$$

As the variable ℓ becomes cyclic in this framework, then the corresponding conjugate variable L is constant.

A more explicit form for the averaged equations of motions (3.23) is

$$\begin{cases} \dot{\widetilde{G}} = \overline{\dfrac{\partial R}{\partial g}} \\ \dot{\widetilde{Z}} = \overline{\dfrac{\partial R}{\partial z}} = 0 \end{cases} \qquad \begin{cases} \dot{\tilde{g}} = -\overline{\dfrac{\partial R}{\partial G}} \\ \dot{\tilde{z}} = -\overline{\dfrac{\partial R}{\partial Z}} \, . \end{cases} \tag{3.24}$$

The solutions of equations (3.23) or (3.24) are representative of the solutions of the full equations of motion if there are no mean motion resonances between the asteroid and the planet and if no close approaches occur between them.

If the derivatives of R with respect to Delaunay's variables are regular functions we can use the *theorem of differentiation under the integral sign* [19] to exchange the derivatives and the integrals in (3.23); then the averaged equations take the form

$$\dot{\widetilde{E}}_{\mathcal{D}} = -\mathcal{J}\, (\nabla_{\mathcal{E}_{\mathcal{D}}} \overline{R})^{\,t}$$

where

$$\overline{R} = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} R\,d\ell\,d\ell' = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\mu k^2}{|x-x'|}\,d\ell\,d\ell' \tag{3.25}$$

(μ is the ratio between the mass of the planet and the mass of the Sun); in fact the average of the indirect part of the perturbation is zero (see Appendix A for a proof of this statement).

Remark 8. From the form of the equations (3.23) we can deduce that if we decrease the perturbation by rescaling the perturbing parameters μ_i, we do not change the evolution trajectories but we only rescale the time needed to that evolution.

3.5.2 An integrable problem

The averaged problem (3.23) has been studied in [78] assuming that all the orbits of the planets in the model are circular and coplanar, following the example of [39].

This assumption makes the averaged dynamics integrable as the problem has three degrees of freedom and we have

Proposition 5. *The following three quantities are first integrals independent and in involution:*

1. *the semimajor axis a;*
2. *the third component of the angular momentum $Z = k\sqrt{a(1-e^2)}\cos I$;*
3. *the averaged perturbing function*

$$\overline{R} = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} R\,d\ell\,d\ell' \quad \textit{(}\text{Kozai integral}\textit{)}. \tag{3.26}$$

Proof. The functional independence of these three quantities is evident; we shall prove that they are in involution. We use the symbol $\{,\}$ for the Poisson brackets; that is, given two functions $f_1(\ell,g,z,L,G,Z)$ and $f_2(\ell,g,z,L,G,Z)$ we have

$$\{f_1,f_2\} = \left(\frac{\partial f_1}{\partial \ell}\frac{\partial f_2}{\partial L} - \frac{\partial f_1}{\partial L}\frac{\partial f_2}{\partial \ell}\right) + \left(\frac{\partial f_1}{\partial g}\frac{\partial f_2}{\partial G} - \frac{\partial f_1}{\partial G}\frac{\partial f_2}{\partial g}\right) + \left(\frac{\partial f_1}{\partial z}\frac{\partial f_2}{\partial Z} - \frac{\partial f_1}{\partial Z}\frac{\partial f_2}{\partial z}\right).$$

As $a = L^2/k^2$ it follows immediately that $\{a,Z\} = 0$. Furthermore we have

$$\{a,\overline{R}\} = -\frac{\partial a}{\partial L}\frac{\partial \overline{R}}{\partial \ell} = 0$$

because we have eliminated the dependence on the mean anomaly ℓ in $\overline{R}$ by averaging over it.

We evaluate the last Poisson bracket:

$$\{Z,\overline{R}\} = -\frac{\partial Z}{\partial Z}\frac{\partial \overline{R}}{\partial z} = 0$$

because the perturbing function itself, even without averaging, does not depend on the longitude of the Node z if we consider the planets on circular coplanar orbits.

□

REMARK 9. Note that the assumption on the symmetry of the orbits of the planets (circular orbits) is essential in order to be able to integrate the problem.

We also observe that Lidov and Ziglin [43] made an analytical study of the restricted circular twice-averaged three body problem Sun-planet-asteroid in the case in which the orbit of the asteroid is uniformly close to the orbit of the planet.

Chapter 4

Averaging on planet crossing orbits: circular coplanar case

We take into account the possibility of intersections between the orbit of an asteroid an the orbits of the planets in the framework of a restricted problem. Let us give the following

DEFINITION 13. *We say that an asteroid is* planet crossing *if its orbit crosses the orbit of some planet during its evolution.*

Starting from 1898, the year in which the asteroid Eros (433) was discovered, a new asteroid population has been detected: the *Near Earth Asteroids* (NEAs). We say that an asteroid is a NEA if its osculating perihelion q satisfies the relation

$$q = a(1-e) < 1.3AU \ .$$

In the last years, thanks to the new observational techniques, the number of known NEAs has increased very rapidly, and up to now we know the existence of about 1800 of such asteroids.

As the orbits of the NEAs are close to the one of the Earth and of the inner planets, generally these asteroids are planet crossing.

4.1 Difficulties arising with planet crossing orbits

When we consider a planet crossing asteroid at the time of intersection of the orbits, the averaged perturbing function $\overline{R}$ is the integral of an unbounded function that is convergent because $1/|x-x'|$ has a first order polar singularity in the values $\overline{\ell}, \overline{\ell}'$ corresponding to a collision. The derivatives at the right hand side of (3.23) have second order polar singularities in $\overline{\ell}, \overline{\ell}'$, hence equations (3.23) do not make sense in this case because the integrals over ℓ, ℓ' of these derivatives are divergent and the classical averaging principle cannot be applied.

The study of the dynamics of NEAs requires the development of theories that can give statistical informations on the evolution of these objects, because the evolution starting from a single initial condition is not representative for long time spans, due to the chaoticity of the orbits; thus it is desirable to have an averaging method also in this case.

In this chapter we describe a generalization of the classical averaging principle, first presented in [28], that is suitable for planet crossing orbits in the framework of a Solar System with the planets on circular and coplanar orbits. We shall refer to this problem as to the *circular coplanar case*.

4.2 Geometry of the node crossing

We give a description of the possible geometric configurations in the plane $(e\cos\omega, e\sin\omega)$ when a node crossing occurs.

Recall that the ascending and descending node crossings with a planet with semimajor axis a' are characterized by the vanishing of the following expressions respectively:

$$d^+_{nod} = \frac{a(1-e^2)}{1+e\cos\omega} - a' ; \qquad d^-_{nod} = \frac{a(1-e^2)}{1-e\cos\omega} - a' ;$$

that are called *nodal distances* (and can be negative).

As we observed in Chapter 3, in the averaged problem with the planets on circular coplanar orbits we have three integrals of motions: the semimajor axis a, the *Kozai integral* $\overline{H} = H_0 - \overline{R}$ and the *z-component of the angular momentum* $Z = k\sqrt{a(1-e^2)}\cos I$. The Z integral allows to determine the evolution of the averaged inclination $I(t)$ if we know $e(t)$; if we also know the evolution of the averaged perihelion argument $\omega(t)$ we can determine $\Omega(t)$ by a simple quadrature of $\partial\overline{R}/\partial Z$, that does not depend on Ω. From the expression of the integral Z we deduce the maximum value of the averaged inclination and eccentricity:

$$I_{max} = I|_{e=0} = \arccos\frac{Z}{k\sqrt{a}} ; \qquad e_{max} = e|_{I=0} = \frac{\sqrt{k^2a - Z^2}}{k\sqrt{a}} .$$

For a given value of the semimajor axis a we can represent the level lines of the averaged Hamiltonian, on which the averaged solutions evolve, in the plane $(\xi,\eta) := (e\cos\omega, e\sin\omega)$. We define the *Kozai domain*

$$W = \{(\xi,\eta) : \xi^2 + \eta^2 \le e^2_{max}\} ,$$

where the averaged dynamics is confined.

In the (ξ,η) reference plane the node crossing lines with the planets are circles: they are defined by

$$\Gamma^+(i) = \{(\xi,\eta) : d^+_{nod}(i) = 0\} ; \qquad \Gamma^-(i) = \{(\xi,\eta) : d^-_{nod}(i) = 0\} ;$$

where i is the index of the planet.

At the ascending node crossing with the planet i the equation to be considered is

$$1 - \xi^2 - \eta^2 = \frac{a'_i}{a}(1+\xi) .$$

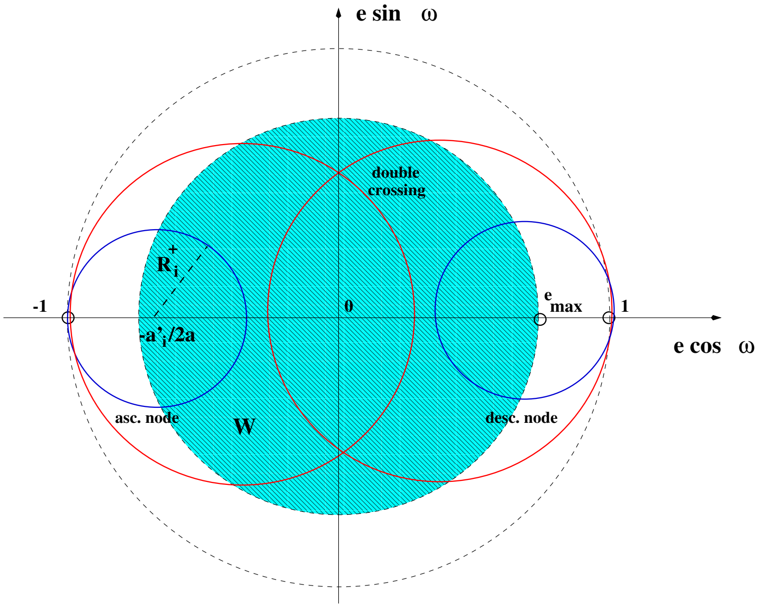

Figure 4.1: The Kozai domain $\{(e\cos\omega, e\sin\omega) : 0 \leq e \leq e_{max}, \omega \in \mathbb{R}\}$ is represented in the figure by the set W. We plot also the four circles corresponding to ascending and descending node crossing with two planets (they have their centers shifted respectively on the left and on the right). An additional exterior circle corresponding to the boundary for closed orbits ($e = 1$) is drawn.

After the coordinate change $X = \xi + a_i'/(2a)$; $Y = \eta$ we obtain

$$X^2 + Y^2 = \left(1 - \frac{a_i'}{2a}\right)^2 ,$$

that is, in the (ξ, η) plane, the equation of a circle of radius $R_i^+ = 1 - a_i'/(2a)$, with center in $(\xi_i^+, \eta_i^+) = (-a_i'/(2a), 0)$ (see Figure 4.1).

By the previous calculations we have

$$\begin{cases} d_{nod}^+(i) > 0 & \text{inside } \Gamma^+(i) \\ d_{nod}^+(i) < 0 & \text{outside } \Gamma^+(i) \ . \end{cases}$$

In a similar way we can prove that the equation $d_{nod}^-(i) = 0$ represents a circle of radius $R_i^- = R_i^+ = 1 - a_i'/(2a)$, with center in $(\xi_i^-, \eta_i^-) = (+a_i'/(2a), 0)$; furthermore we have

$$\begin{cases} d_{nod}^-(i) > 0 & \text{inside } \Gamma^-(i) \\ d_{nod}^-(i) < 0 & \text{outside } \Gamma^-(i) \ . \end{cases}$$

Definition 14. *A* double (node) crossing *is a crossing between the orbit of the asteroid and the orbit of a planet at both the ascending and descending node.*

By the symmetry of the circles $\{d^+_{nod}(i) = 0\}$ and $\{d^-_{nod}(i) = 0\}$, for each index i, we can deduce that a double crossing is possible only when $\omega = \pi/2$ or $\omega = 3\pi/2$ (see Figure 4.1). We obtain the following condition on the ratio of the semimajor axes a, a'_i:

$$-\frac{a'_i}{2a} \geq -\frac{1}{2} \qquad \text{that is} \qquad a \geq a'_i \ ,$$

and in particular we obtain that *there cannot exist Athen asteroids* (see Chapter 7) *that have a double crossing with the Earth.*

DEFINITION 15. *A* simultaneous crossing *is a crossing of the orbit of the asteroid and the orbits of two planets at the same time.*

We note that if we call a'_1, a'_2 the semimajor axes of the orbits of two different planets, we cannot have a simultaneous crossing at the ascending node of both planets (this would imply $a'_1 = a'_2$ in this model). By a similar argument we cannot have a simultaneous crossing at the descending node of both planets. On the other hand we have a simultaneous crossing at the ascending node with one planet and at descending node with the other one if

$$a'_1 = \frac{a(1-e^2)}{1+e\cos\omega} \qquad \text{and} \qquad a'_2 = \frac{a(1-e^2)}{1-e\cos\omega} \ ,$$

that is if

$$\frac{a'_1}{a'_2} = \frac{1-e\cos\omega}{1+e\cos\omega} \ .$$

In this framework we cannot have crossings of different type from the ones presented above (like *triple crossing*, etc.).

4.3 Description of the osculating orbits

We consider a model with three bodies only: Sun, planet, asteroid. We set the x_1 axis along the line of the nodes, pointing towards the ascending mutual node, and the x_2 axis is chosen on the orbital plane of the planet (see Figure 4.2). The equations defining the osculating orbits $P(u) = (p_1(u), p_2(u), p_3(u))$ and $P'(u') = \big(p'_1(u'), p'_2(u'), p'_3(u')\big)$ of the asteroid and the planet are

$$\begin{cases} p_1 = a[(\cos u - e)\cos\omega - \beta\sin u\sin\omega] \\ p_2 = a[(\cos u - e)\sin\omega + \beta\sin u\cos\omega]\cos I \\ p_3 = a[(\cos u - e)\sin\omega + \beta\sin u\cos\omega]\sin I \end{cases} \qquad \begin{cases} p'_1 = a'\cos u' \\ p'_2 = a'\sin u' \\ p'_3 = 0 \end{cases} \tag{4.1}$$

where u, u' are the eccentric anomalies and $\beta = \sqrt{1-e^2}$. These orbits are respectively an ellipse and a circle.

The distance between a point on an orbit and a point on the other one, appearing at the denominator of the direct term of the perturbing function, is defined by its square as

$$\mathfrak{D}^2(u,u') = (p_1 - p'_1)^2 + (p_2 - p'_2)^2 + (p_3 - p'_3)^2 =$$

$$= a^2(1-e\cos u)^2 + a'^2 - 2aa'\{\cos u'[(\cos u - e)\cos\omega - \beta\sin u \sin\omega] + \\ + \sin u' \cos I[(\cos u - e)\sin\omega + \beta \sin u \cos\omega]\} \; .$$

We introduce the function $\mathrm{D}(\ell,\ell')$, which is implicitly defined by

$$\mathrm{D}\left(\ell(u),\ell'(u')\right) = \mathfrak{D}(u,u')$$

and by Kepler's equations

$$\ell = u - e\sin u\,; \qquad\qquad \ell' = u' \tag{4.2}$$

for the asteroid and the planet (the latter has a simpler form because the orbit of the planet is circular).

We define the values of the anomalies $\overline{u},\overline{u}'$ corresponding to the mutual ascending node: we immediately notice that $\overline{u}' = 0$, while from

$$a(1-e\cos\overline{u}) = \frac{a\beta^2}{1+e\cos\omega}$$

we obtain

$$\cos\overline{u} = \frac{\cos\omega + e}{(1+e\cos\omega)}; \qquad \sin\overline{u} = -\frac{\beta\sin\omega}{(1+e\cos\omega)}$$

(the sign of $\sin\overline{u}$ has been chosen in such a way that it is opposite to the sign of $\sin\omega$).

The equations defining the anomalies $\overline{u}_1,\overline{u}'_1$, corresponding to the mutual descending node, are

$$\overline{u}'_1 = \pi; \qquad \cos\overline{u}_1 = \frac{e-\cos\omega}{(1-e\cos\omega)}; \qquad \sin\overline{u}_1 = \frac{\beta\sin\omega}{(1-e\cos\omega)} \; .$$

In the following we shall study the case of the ascending node crossings, but the same methods are suitable to deal also with the descending ones and with double crossings: we shall give the formulas to be applied in these cases at the end of this chapter and in Appendix B.

4.4 Weak averaged solutions

The idea of the generalization of the averaging principle in [28] starts from the fact that if there are no crossings between the orbits, by the *theorem of differentiation under the integral sign* [19], the averaged equations of motion (3.23) are equivalent to Hamilton's equations (4.3) with the averaged perturbing function $\overline{R}$:

$$\dot{\overline{\mathcal{E}}}_{\mathcal{D}} = -\mathcal{J}\left[\nabla_{\mathcal{E}_{\mathcal{D}}}\overline{R}\right]^t \tag{4.3}$$

where

$$\overline{R} = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} R\, d\ell\, d\ell' \; .$$

We write equations (4.3) in a more explicit form:

$$\begin{cases} \dot{\tilde{G}} = \dfrac{\partial \overline{R}}{\partial g} \\ \dot{\tilde{Z}} = \dfrac{\partial \overline{R}}{\partial z} = 0 \end{cases} \qquad \begin{cases} \dot{\tilde{g}} = -\dfrac{\partial \overline{R}}{\partial G} \\ \dot{\tilde{z}} = -\dfrac{\partial \overline{R}}{\partial Z} \, ; \end{cases} \tag{4.4}$$

We shall skip the 'tilde' over the averaged variables in the following, to avoid the use of a heavy notation.

We shall prove that when the orbits intersect each other it is possible to define piecewise smooth solutions of equations (4.4), that we call *weak averaged solutions*, and we shall see that the loss of regularity corresponds exactly to the crossing configurations of the orbits: in fact we shall give a twofold meaning to the right hand sides of (4.4) at the node crossing, corresponding to the two limit values of the derivatives coming from inside and outside the circle that represents the ascending node crossing with the planet in the plane (ξ,η).

Note that the weak averaged solutions correspond to the classical averaged solutions as far as their trajectories in the reduced phase space (ξ,η) do not pass through a node crossing line.

We also observe that the exchange of the differential and integral operators in (4.4) is not essential for a theoretical definition of the weak solutions (they could anyway be defined as the limits of the solutions of (3.23) coming from both sides of the node crossing lines) but, as we shall see, this operation is necessary to obtain analytic formulas for the discontinuity of the average of the derivatives of R, that are not defined on the node crossing lines, and to define the semianalytic procedure to compute the weak solutions.

4.5 The Wetherill function

Let $\{P(\overline{u}), P'(\overline{u}')\}$ be the ascending mutual node. We consider the two straight lines $\mathrm{r}(\ell)$ and $\mathrm{r}'(\ell')$, tangent in $P(\overline{u})$ and $P'(\overline{u}')$ to the orbits of the asteroid and of the planet (see Figure 4.2); they can be parametrized by the mean anomalies ℓ,ℓ' so that $P(u(t))$ and $\mathrm{r}(\ell(t))$ have the same velocities (derivatives with respect to t) in $P(\overline{u})$ and $P'(u'(t))$ and $\mathrm{r}'(\ell'(t))$ have the same velocities in $P'(\overline{u}')$:

$$\begin{cases} r_1 = \overline{x}_1 - \mathcal{F}k \\ r_2 = \overline{x}_2 + \mathcal{G}\cos I k \\ r_3 = \overline{x}_3 + \mathcal{G}\sin I k \end{cases} \qquad \begin{cases} r_1' = \overline{x}_1' \\ r_2' = \overline{x}_2' + a'k' \\ r_3' = \overline{x}_3' \end{cases} \tag{4.5}$$

where $k = \ell - \overline{\ell}, k' = \ell' - \overline{\ell}'$, and $\overline{\ell},\overline{\ell}'$ are the values of the mean anomalies corresponding to $\overline{u},\overline{u}'$ (so that $\overline{\ell}' = 0$). We have used the following notations

$$\mathcal{F} \;=\; \frac{\sin\overline{u}\cos\omega + \beta\cos\overline{u}\sin\omega}{1 - e\cos\overline{u}} = \frac{ae\sin\omega}{\beta}\,;$$

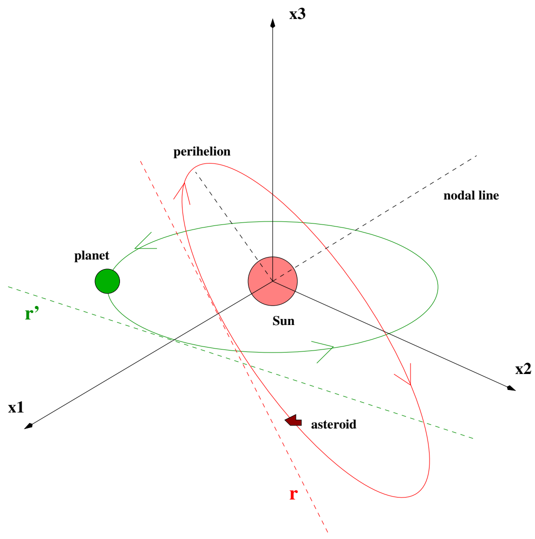

Figure 4.2: The straight lines r, r′ represent Wetherill's approximation at the ascending node for the two osculating orbits of the asteroid and the planet.

$$\mathcal{G} \;=\; \frac{-\sin\overline{u}\sin\omega+\beta\cos\overline{u}\cos\omega}{1-e\cos\overline{u}}=\frac{a(1+e\cos\omega)}{\beta}$$

and

$$\overline{x}_1=\frac{a\beta^2}{1+e\cos\omega}; \qquad \overline{x}'_1=a'; \qquad \overline{x}_2=\overline{x}_3=\overline{x}'_2=\overline{x}'_3=0\;.$$

Remark 10. Taylor's series developments of equations (4.1) in a neighborhood of the points $\overline{u},\overline{u}'$ correspond, up to the first order in $v=u-\overline{u}, v'=u'-\overline{u}'$, to the Taylor's series developments in the same variables of equations (4.5):

$$\begin{cases} p_1=a\beta^2(1+e\cos\omega)^{-1}-\mathcal{F}(1-e\cos\overline{u})\,v+o(v)\\ p_2=\mathcal{G}\cos I(1-e\cos\overline{u})\,v+o(v)\\ p_3=\mathcal{G}\sin I(1-e\cos\overline{u})\,v+o(v)\end{cases} \qquad \begin{cases} p'_1=a'\\ p'_2=a'v'\\ p'_3=0\end{cases} \tag{4.6}$$

From Kepler's equations we deduce

$$v=k(1-e\cos\overline{u})^{-1}+O(k)$$

in a neighborhood of $k=0$, so that

$$\begin{cases} p_1=a\beta^2(1+e\cos\omega)^{-1}-\mathcal{F}\,k+O(k^2)\\ p_2=\mathcal{G}\cos I\,k+O(k^2)\\ p_3=\mathcal{G}\sin I\,k+O(k^2)\;.\end{cases}$$

DEFINITION 16. *We call* Wetherill function *[77] the approximated distance function* d, *whose square is defined by*

$$\begin{aligned}\mathrm{d}^2(\ell,\ell') &= (r_1-r_1')^2+(r_2-r_2')^2+(r_3-r_3')^2 = \\ &= a'^2k'^2+[\mathcal{F}^2+\mathcal{G}^2]k^2-2kk'[\mathcal{G}a'\cos I]-2d^+_{nod}\mathcal{F}k+(d^+_{nod})^2\end{aligned}$$

with $k=\ell-\overline{\ell}, k'=\ell'$.

Note that d^2 is a quadratic form in the variables k,k': it is homogeneous when there is a crossing at the ascending node. We can write it more concisely as

$$\mathrm{d}^2(\ell,\ell') = d^2(\kappa) = \kappa^t\mathcal{A}\kappa + B^t\kappa + (d^+_{nod})^2$$

where

$$\kappa = (k',k)\ ; \qquad B = 2(B_1,B_2)\ ; \qquad \mathcal{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{bmatrix}\ ;$$

with components

$$\begin{cases} B_1 = 0 \\ B_2 = -d^+_{nod}\mathcal{F}\ ; \end{cases} \qquad \begin{cases} A_{11} = a'^2 \\ A_{12} = A_{21} = -\mathcal{G}a'\cos I \\ A_{22} = [\mathcal{F}^2+\mathcal{G}^2]\ . \end{cases}$$

For later use we define

$$\mathfrak{d}^2(u,u') = \mathrm{d}^2(\ell(u),\ell'(u'))\ .$$

The geometry of Wetherill's straight lines is strictly related to the degeneracy of the matrix $\mathcal{A}$, in fact we have

LEMMA 1. *The matrix* $\mathcal{A}$ *is always* positive definite *if* $I>0$. *If* $I=0$ *we have degeneracy of* $\mathcal{A}$ *if and only if the straight lines* r, r′ *are parallel: in this case* $\mathcal{A}$ *is* positive semi-definite.

Proof. $\mathcal{A}$ is a symmetric 2×2 matrix and it is positive definite if and only if its principal invariants, the trace $tr(\mathcal{A})$ and the determinant $det(\mathcal{A})$, are positive. By a direct computation we have

$$\begin{cases} tr(\mathcal{A}) = a'^2+\mathcal{F}^2+\mathcal{G}^2 \\ \\ det(\mathcal{A}) = a'^2\left(\mathcal{F}^2+\mathcal{G}^2\sin^2 I\right)\ . \end{cases}$$

From the above expressions we deduce that $tr(\mathcal{A})>0$ (we are considering only bounded orbits, so that $0\le e<1$); furthermore

$$det(\mathcal{A}) = 0 \quad \Longleftrightarrow \quad \begin{cases} I = 0 \\ e\sin\omega = 0\ , \end{cases}$$

that corresponds to the straight lines r, r′ being parallel. □

Definition 17. *We call* tangent crossings *the crossing orbital configurations for which* $det(\mathcal{A}) = 0$.

The assumption that the inclination I of the asteroid is different from zero during its whole time evolution, or at least in a neighborhood of each crossing between the orbits, implies that *no tangent crossings occur*.

4.6 Kantorovich's method of singularity extraction

We shall describe Kantorovich's method of singularity extraction (see [15]) that allows to improve the stability of the numerical computation of the integrals when the integrand function $f_1(x)$ is unbounded in the neighborhood of one or more points.

Kantorovich's method consists in searching for a function $f_2(x)$ whose primitive has an analytic expression in terms of elementary functions and such that the difference $f_1(x) - f_2(x)$ is more regular than $f_1(x)$ (for example it is bounded or even continuous).

It is then convenient to split the computation as follows

$$\int f_1(x)\,dx = \int [f_1(x) - f_2(x)]\,dx + \int f_2(x)\,dx$$

so that the singularity has moved to the second term, that can be better handled.

This method can help us to study the regularity properties of the averaged perturbing function $\overline{R}$ defined in (3.25); we shall use the inverse of the Wetherill function $1/\mathrm{d}$ to extract the principal part from the direct term of the perturbing function.

The function D is 2π-periodic in both variables ℓ, ℓ' and this property can be used to shift the integration domain

$$\mathbb{T}^2 = \{(\ell, \ell') : -\pi \le \ell \le \pi, -\pi \le \ell' \le \pi\}$$

in a suitable way, so that the crossing values $(\overline{\ell}, 0)$ will be always internal points of this domain.

We shall prove that in computing the derivatives of $\overline{R}$ with respect to the variables $\mathcal{E}_{\mathcal{D}}$, for instance the G-derivative, we can use the decomposition

$$\frac{(2\pi)^2}{\mu k^2}\frac{\partial}{\partial G}\overline{R} = \int_{\mathbb{T}^2} \frac{\partial}{\partial G}\left[\frac{1}{\mathrm{D}} - \frac{1}{\mathrm{d}}\right] d\ell\, d\ell' + \frac{\partial}{\partial G}\int_{\mathbb{T}^2} \frac{1}{\mathrm{d}}\, d\ell\, d\ell'; \tag{4.7}$$

namely we shall prove the validity of the hypotheses of the theorem of differentiation under the integral sign to exchange the symbols of integral and derivative in front of the *remainder function* $1/\mathrm{D} - 1/\mathrm{d}$. The average of the remainder function is then differentiable as it is derivable with continuity with respect to all the variables $\mathcal{E}_{\mathcal{D}}$. Therefore we shall need only to study the regularity properties of the last term of the sum in (4.7), which is easier to handle.

Note that we use Kantorovich's method of singularity extraction in a wider extent: the derivatives of the remainder function still have a polar singularity in $(\overline{\ell}, 0)$, but it is of order one, so that the integrals over ℓ, ℓ' of these derivatives are convergent.

4.7 Integration of $1/\mathrm{d}$

We shall discuss the analytic method to integrate $1/\mathrm{d}$ over the torus $\mathbb{T}^2 = \{(\ell,\ell') : -\pi \leq \ell \leq \pi, -\pi \leq \ell' \leq \pi\}$ assuming that $(\overline{\ell},0)$ is an internal point of this domain.

We move the ascending node crossing point $(\overline{\ell},0)$ to the origin of the reference system by the variable change

$$\tau_{\overline{\ell},0} : \ (\ell,\ell') \longrightarrow (k,k') = (\ell - \overline{\ell},\ell') \tag{4.8}$$

and we set

$$\overline{\mathbb{T}}^2 = \tau_{\overline{\ell},0}\left[\mathbb{T}^2\right] = \{(\ell-\overline{\ell},\ell') : (\ell,\ell') \in \mathbb{T}^2\}\ .$$

Then we perform another variable change to eliminate the linear terms in the quadratic form $d^2(\kappa)$ defined by (4.7). The inverse of the transformation used for this purpose is

$$\Xi^{-1} : \psi \longrightarrow \kappa = \mathcal{T}\,\psi + S \tag{4.9}$$

where $S = (S_1,S_2) \in \mathbb{R}^2,\ \psi = (y',y) \in \mathbb{R}^2$ are the new variables and $\mathcal{T}$ is a 2×2 real-valued invertible matrix.

Setting to zero the coefficients of the linear terms of the quadratic form in the new variables ψ we obtain the equations

$$2\,\mathcal{A}S + B = 0$$

whose solutions are

$$S_1 = \frac{B_2A_{12}}{det(\mathcal{A})} = \frac{d^+_{nod}\mathcal{F}\,\mathcal{G}a'\cos I}{det(\mathcal{A})}\ ;$$

$$S_2 = -\frac{B_2A_{11}}{det(\mathcal{A})} = \frac{d^+_{nod}\mathcal{F}\,a'^2}{det(\mathcal{A})}\ .$$

We can determine the non-degenerate matrix $\mathcal{T}$ in order to obtain

$$\mathcal{T}^t\mathcal{A}\mathcal{T} = I_2 \tag{4.10}$$

where I_2 is the 2×2 identity matrix: from relation (4.10) we obtain $\mathcal{A} = \left(\mathcal{T}^{-1}\right)^t\mathcal{T}^{-1}$; let us set

$$\mathcal{R} = \mathcal{T}^{-1} = \begin{pmatrix} \rho_1 & \rho_2 \\ \rho_3 & \rho_4 \end{pmatrix}\ ; \tag{4.11}$$

we obtain the system

$$\begin{cases} \rho_1^2 + \rho_3^2 = A_{11} \\ \rho_1\rho_2 + \rho_3\rho_4 = A_{12} \\ \rho_2^2 + \rho_4^2 = A_{22}\ . \end{cases} \tag{4.12}$$

System (4.12) has infinitely many solutions; we notice that the relations $A_{11} > 0$ and $det(\mathcal{A}) > 0$ allows us to find four solutions $(\rho_1,\rho_2,0,\rho_4)$ with $\rho_3 = 0$:

$$\rho_1 = \pm\sqrt{A_{11}}; \qquad \rho_2 = \frac{A_{12}}{\rho_1}; \qquad \rho_4 = \pm\sqrt{\frac{det(\mathcal{A})}{A_{11}}}\ . \tag{4.13}$$

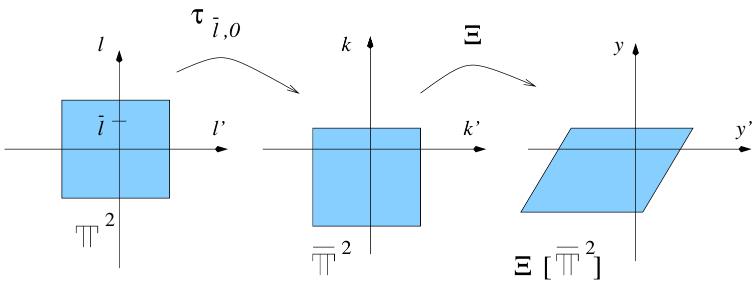

Figure 4.3: Description of the transformations of the integration domain $\mathbb{T}^2$ with the two coordinate changes (4.8), (4.9) used to bring the squared Wetherill function $\mathrm{d}^2(\ell,\ell')$ into the form $y^2+y'^2+(d^+_{min})^2$ in the new variables (y',y). Note that $\overline{\ell}'=0$ implies that $\overline{\mathbb{T}}^2$ is symmetric with respect to the k axis.

We select the signs in the above expressions and we call $(\tau,\sigma,0,\rho)$ the particular solution defined by

$$\tau = a'; \qquad \sigma = -\mathcal{G}\cos I; \qquad \rho = \frac{1}{a'}\sqrt{det(\mathcal{A})}; \tag{4.14}$$

so that we can write

$$\mathcal{T} = \frac{1}{\sqrt{det(\mathcal{A})}}\begin{pmatrix} \rho & -\sigma \\ 0 & \tau \end{pmatrix} = \begin{pmatrix} 1/\tau & -\sigma/\tau\rho \\ 0 & 1/\rho \end{pmatrix} . \tag{4.15}$$

The coordinate change

$$\Xi : \kappa \longrightarrow \psi = \mathcal{R}\left[\kappa - S\right] , \tag{4.16}$$

where

$$\mathcal{R} = \mathcal{T}^{-1} = \begin{pmatrix} \tau & \sigma \\ 0 & \rho \end{pmatrix} ,$$

brings $d^2(\kappa)$ into the form

$$d^2\left(\Xi^{-1}(\psi)\right) = y^2 + y'^2 + (d^+_{min})^2$$

in the new variables ψ, with

$$d^+_{min} = |d^+_{nod}|\left\{1 - \frac{a'^2\mathcal{F}^2}{det(\mathcal{A})}\right\}^{1/2} = |d^+_{nod}|\left[\frac{a'^2\mathcal{G}^2\sin^2 I}{det(\mathcal{A})}\right]^{1/2} . \tag{4.17}$$

The domain $\overline{\mathbb{T}}^2$ is transformed into a parallelogram with two sides parallel to the y' axis (see Figure 4.3).

Remark 11. Note that d^+_{min} is the minimal distance between the straight lines r and r' (see also the remark after equation (9.16) in Chapter 9).

Using the variable changes (4.8), (4.16) and the transformation to polar coordinates, whose inverse is

$$\Pi^{-1} : \begin{pmatrix} r \\ \theta \end{pmatrix} \longrightarrow \begin{pmatrix} y' \\ y \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$

we obtain

$$\begin{aligned} \int_{\mathbb{T}^2} \frac{1}{\mathrm{d}} d\ell\, d\ell' &= \frac{1}{\sqrt{det(\mathcal{A})}} \int_{\Xi\left[\overline{\mathbb{T}}^2\right]} \frac{1}{\sqrt{y^2 + y'^2 + (d^+_{min})^2}} dy\, dy' = \\ &= \frac{1}{\sqrt{det(\mathcal{A})}} \int_{\mathfrak{T}} \frac{r}{\sqrt{r^2 + (d^+_{min})^2}} dr\, d\theta \end{aligned} \tag{4.18}$$

where $\Xi^{-1}\left[\Pi^{-1}(\mathfrak{T})\right] = \overline{\mathbb{T}}^2$.

Let us describe the domain $\mathfrak{T}$ in details. We define the straight lines that bound the integration domain $\Xi\left[\overline{\mathbb{T}}^2\right]$ as

$$\begin{aligned} \mathrm{r}_1 &= \{(y,y') : y' = \frac{\sigma}{\rho} y + \tau(\pi - S_1)\}\,; \qquad \mathrm{r}_2 = \{(y,y') : y = \rho(\pi - \overline{\ell} - S_2)\}\,; \\ \mathrm{r}_3 &= \{(y,y') : y' = \frac{\sigma}{\rho} y - \tau(\pi + S_1)\}\,; \qquad \mathrm{r}_4 = \{(y,y') : y = -\rho(\pi + \overline{\ell} + S_2)\}\,. \end{aligned}$$

The intersections of these lines with the y axis are

$$y_1 = -\rho(\pi + \overline{\ell} + S_2)\,; \qquad y_2 = \rho(\pi - \overline{\ell} - S_2)\,;$$

while the intersections with the y' axis are

$$\begin{aligned} y'_1 &= \lambda_3(y_1) = -\sigma(\pi + \overline{\ell} + S_2) - \tau(\pi + S_1) \\ y'_2 &= \lambda_3(0) = -\tau(\pi + S_1) \\ y'_3 &= \lambda_3(y_2) = \sigma(\pi - \overline{\ell} - S_2) - \tau(\pi + S_1) \\ y'_4 &= \lambda_1(y_1) = -\sigma(\pi + \overline{\ell} + S_2) + \tau(\pi - S_1) \\ y'_5 &= \lambda_1(0) = \tau(\pi - S_1) \\ y'_6 &= \lambda_1(y_2) = \sigma(\pi - \overline{\ell} - S_2) + \tau(\pi - S_1) \end{aligned}$$

where $\lambda_1(y) = (\sigma/\rho)y + \tau(\pi - S_1)$ and $\lambda_3(y) = (\sigma/\rho)y - \tau(\pi + S_1)$.

We can then decompose the domain $\mathfrak{T}$ into four parts (see Figure 4.4)

$$\mathfrak{T} = \bigcup_{j=1}^{4} \left\{(r,\theta) \in \mathbb{R}^2 \ : \ \theta_j \le \theta \le \theta_{j+1} \text{ and } 0 \le r \le r_j(\theta)\right\}$$

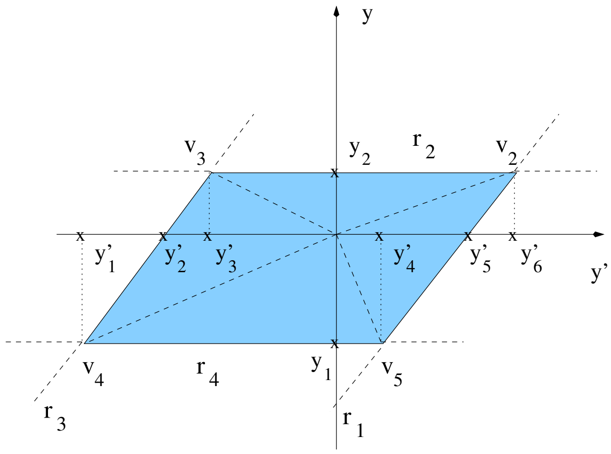

Figure 4.4: We show the decomposition of the integration domain used to compute the last integral in (4.18) in polar coordinates.

where $r_j(\theta)$, with $j = 1\dots4$, represent the lines r_j delimiting $\Xi[\overline{\mathbb{T}}^2]$ in polar coordinates:

$$r_1(\theta) = \frac{\rho\tau(\pi - S_1)}{\rho\cos\theta - \sigma\sin\theta}; \qquad r_2(\theta) = \frac{\rho(\pi - \overline{\ell} - S_2)}{\sin\theta};$$

$$r_3(\theta) = \frac{-\rho\tau(\pi + S_1)}{\rho\cos\theta - \sigma\sin\theta}; \qquad r_4(\theta) = \frac{-\rho(\pi + \overline{\ell} + S_2)}{\sin\theta};$$

while $\theta_1 = \theta_5 - 2\pi$ and θ_l, with $l = 2\dots5$, are the counter-clockwise angles between the y'-axis and the vertexes v_l seen from the origin of the axes (see Figure 4.4):

$$0 < \theta_2 < \theta_3 < \pi < \theta_4 < \theta_5 < 2\pi;$$

$$\tan\theta_2 = \frac{\rho(\pi - \overline{\ell} - S_2)}{\sigma(\pi - \overline{\ell} - S_2) + \tau(\pi - S_1)}; \quad \tan\theta_3 = \frac{\rho(\pi - \overline{\ell} - S_2)}{\sigma(\pi - \overline{\ell} - S_2) - \tau(\pi + S_1)};$$

$$\tan\theta_4 = \frac{\rho(\pi + \overline{\ell} + S_2)}{\sigma(\pi + \overline{\ell} + S_2) + \tau(\pi + S_1)}; \quad \tan\theta_5 = \frac{\rho(\pi + \overline{\ell} + S_2)}{\sigma(\pi + \overline{\ell} + S_2) - \tau(\pi - S_1)}.$$

Using the previous decomposition for $\mathfrak{T}$ and integrating in the r variable the last expression in (4.18) we obtain

$$\int_{\mathbb{T}^2} \frac{1}{\mathrm{d}}\, d\ell\, d\ell' = \frac{1}{\sqrt{det(\mathcal{A})}} \cdot \left\{ \sum_{j=1}^{4} \int_{\theta_j}^{\theta_{j+1}} \sqrt{(d^+_{min})^2 + r_j^2(\theta)}\, d\theta - 2\pi d^+_{min} \right\}. \tag{4.19}$$

Note that the integrals in (4.19) are elliptic and the integrand functions are bounded so that these integrals are differentiable functions of the orbital elements. We shall see that the loss of regularity of the averaged perturbing function is due only to the term d^+_{min}.

4.8 Boundedness of the remainder function

When there is a crossing at the ascending node, then from the equations of the orbits (4.1) and from Kepler's equations (4.2) we deduce that Taylor's development of $\mathcal{D}^2(\kappa) = \mathrm{D}^2(\ell,\ell')$ in a neighborhood of $\kappa = (0,0)$ is given by

$$\mathcal{D}^2(\kappa) = d^2(\kappa) + O(|\kappa|^3) \tag{4.20}$$

where $O(|\kappa|^3)$ is an infinitesimal of the same order as $|\kappa|^3$ for $|\kappa| \to 0$. We prove the following

LEMMA 2. *If there is an ascending node crossing between the orbits, there exist a neighborhood $\mathcal{U}_0$ of $\kappa = (0,0)$ and two positive constants B_1, B_2 such that*

$$B_1 d^2(\kappa) \leq \mathcal{D}^2(\kappa) \leq B_2 d^2(\kappa) \qquad \forall \kappa \in \mathcal{U}_0 \, .$$

Proof. First we notice that for $d^+_{nod} = 0$ we have $d^2(\kappa) = \kappa^t \mathcal{A} \kappa$, where $\mathcal{A}$ is positive definite, hence there exist two positive constants C_1, C_2 such that

$$C_1 |\kappa|^2 \leq \kappa^t \mathcal{A} \kappa \leq C_2 |\kappa|^2 \qquad \forall \kappa \in \mathbb{R}^2. \tag{4.21}$$

Using the relations (4.20) and (4.21) we obtain

$$\lim_{|\kappa| \to 0} \frac{\mathcal{D}^2(\kappa)}{d^2(\kappa)} = 1 \, ,$$

that implies the existence of the neighborhood $\mathcal{U}_0$ and of the constants B_1, B_2 as in the statement of the lemma. □

We prove the following result:

PROPOSITION 6. *The remainder function* $1/\mathrm{D} - 1/\mathrm{d}$ *is bounded even if there is an ascending node crossing.*

Proof. If there are no crossings between the orbits the remainder function is trivially bounded, in fact $\mathrm{D}(\ell,\ell') > 0$ for each $(\ell,\ell') \in \mathbb{T}^2$ and the minimal value of $\mathrm{d}(\ell,\ell')$ is d^+_{min} that, for $I \neq 0$, can be zero only if $d^+_{nod} = 0$ (see equation (4.17)).

If there is a crossing at the ascending node we have to investigate the local behavior of the remainder function in a neighborhood of $(\ell,\ell') = (\bar{\ell},0)$, where both D and d can vanish. The boundedness of the remainder function can be shown using the previous lemma: we know that there exists a neighborhood $\mathcal{U}_0$ and a positive constant B_1 such that the relation

$$\mathcal{D}(\kappa) \geq \sqrt{B_1} d(\kappa)$$

holds for each $\kappa \in \mathcal{U}_0$. It follows that in this neighborhood the remainder function can be bounded in the following way:

$$\begin{aligned}\left|\frac{1}{\mathcal{D}(\kappa)} - \frac{1}{d(\kappa)}\right| &= \frac{|d^2(\kappa) - \mathcal{D}^2(\kappa)|}{d(\kappa)\mathcal{D}(\kappa)[d(\kappa)+\mathcal{D}(\kappa)]} \leq \\ &\leq \frac{1}{\sqrt{B_1}[1+\sqrt{B_1}]} \cdot \frac{|d^2(\kappa) - \mathcal{D}^2(\kappa)|}{|\kappa|^3} \cdot \frac{|\kappa|^3}{d^3(\kappa)} .\end{aligned}$$

We observe that $|d^2(\kappa) - \mathcal{D}^2(\kappa)| = O(|\kappa|^3)$ and that by (4.21) there is a positive constant C_1 such that $d^2(\kappa) \geq C_1|\kappa|^2$, so that there exists a constant $L > 0$ such that

$$\left|\frac{1}{\mathcal{D}(\kappa)} - \frac{1}{d(\kappa)}\right| \leq L \qquad \forall \kappa \in \mathcal{U}_0 .$$

□

REMARK 12. Although the remainder function $1/\mathrm{D} - 1/\mathrm{d}$ is bounded, it is not continuous in $(\ell, \ell') = (\overline{\ell}, 0)$ when there is a crossing at the ascending node; this can be seen, for instance, by computing the limits of this function along the straight lines $k' = \lambda k$ ($\lambda \in \mathbb{R}$) as $k \to 0$.

4.9 Formulas for descending node crossings

Let us assume that a descending node crossing occurs between the orbits, we can follow a similar procedure to define weak averaged solutions of equations (4.4).

We use the same reference frame as for the ascending crossing case, with the x-axis towards the mutual ascending node, in order to be able to use the same formulas also in case of double crossings.

We already observed in Section 4.3 that the values of the eccentric anomalies $\overline{u}_1, \overline{u}'_1$ corresponding to the descending node crossing are given by

$$\overline{u}'_1 = \pi; \qquad \cos\overline{u}_1 = \frac{e - \cos\omega}{1 - e\cos\omega}; \qquad \sin\overline{u}_1 = \frac{\beta\sin\omega}{1 - e\cos\omega};$$

we define $\overline{\ell}_1 = \overline{u}_1 - e\sin\overline{u}_1, \overline{\ell}'_1 = \pi$ as the corresponding values of the mean anomalies.

Let $\{P(\overline{u}_1), P'(\overline{u}'_1)\}$ be the descending mutual node. We consider the two straight lines $\mathrm{r}_1(\ell)$ and $\mathrm{r}'_1(\ell')$, tangent in $P(\overline{u}_1)$ and $P'(\overline{u}'_1)$ to the orbits of the asteroid and of the planet (see Figure 4.5); they can be parametrized by the mean anomalies ℓ_1, ℓ'_1 so that $P(u(t))$ and $\mathrm{r}_1(\ell(t))$ have the same velocities (derivatives with respect to t) in $P(\overline{u}_1)$ and $P'(u'(t))$ and $\mathrm{r}'_1(\ell'(t))$ have the same velocities in $P'(\overline{u}'_1)$:

$$\begin{cases} r_{1,1} = \overline{x}_{1,1} - \mathcal{F}_1 k_1 \\ r_{1,2} = \overline{x}_{1,2} + \mathcal{G}_1 \cos I k_1 \\ r_{1,3} = \overline{x}_{1,3} + \mathcal{G}_1 \sin I k_1 \end{cases} \qquad \begin{cases} r'_{1,1} = \overline{x}'_{1,1} \\ r'_{1,2} = \overline{x}'_{1,2} - a' k'_1 \\ r'_{1,3} = \overline{x}'_{1,3} . \end{cases} \tag{4.22}$$

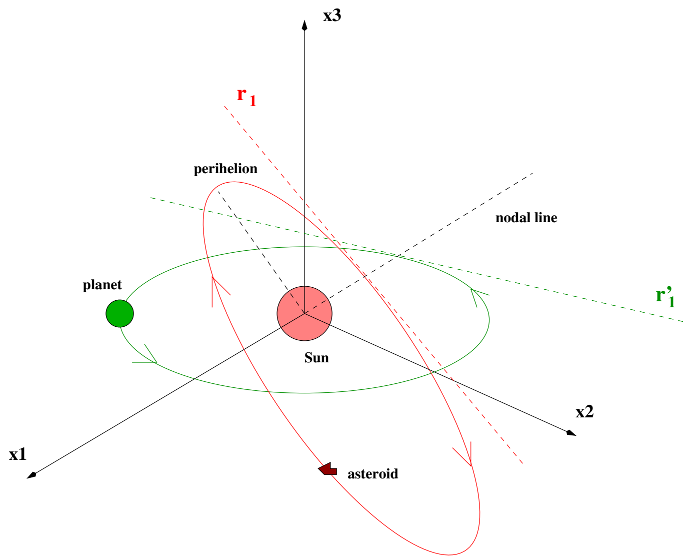

Figure 4.5: The straight lines $\mathrm{r}_1, \mathrm{r}_1'$ represent Wetherill's approximation at the descending node for the two osculating orbits of the asteroid and the planet.

We have used the following notations

$$\begin{aligned}\mathcal{F}_1 &= \frac{\sin\overline{u}_1\cos\omega + \beta\cos\overline{u}_1\sin\omega}{1-e\cos\overline{u}_1} = \frac{ae\sin\omega}{\beta} = \mathcal{F}\,;\\ \mathcal{G}_1 &= \frac{-\sin\overline{u}_1\sin\omega + \beta\cos\overline{u}_1\cos\omega}{1-e\cos\overline{u}_1} = -\frac{a(1-e\cos\omega)}{\beta}\,;\end{aligned}$$

and

$$\overline{x}_{1,1} = -\frac{a\beta^2}{1-e\cos\omega}; \qquad \overline{x}'_{1,1} = -a'; \qquad \overline{x}_{1,2} = \overline{x}_{1,3} = \overline{x}'_{1,2} = \overline{x}'_{1,3} = 0;$$

We can define a *Wetherill function*, similar to the one of the ascending node crossing case, by using straight lines tangent to the orbits at the descending mutual node. The square of this Wetherill function is given by

$$d_1^2 = a'^2 k_1'^2 + [\mathcal{F}_1^2 + \mathcal{G}_1^2]k_1^2 + 2k_1k_1'[\mathcal{G}_1 a'\cos I] + 2k_1 d_{nod}^- \mathcal{F}_1 + (d_{nod}^-)^2$$

where $k_1 = \ell - \overline{\ell}_1, k_1' = \ell' - \pi$ and

$$d_{nod}^- = \frac{a(1-e^2)}{1-e\cos\omega} - a'\,.$$

Remark 13. There are some differences between the general appearance of this formula and the one for the ascending node: they are due to the sign of the mixed quadratic monomial and of the linear term. The positive signs are due, respectively, to the minus sign in the expression of $r'_{1,2}$ and to the fact that $\overline{x}_{1,1} - \overline{x}'_{1,1} = -d_{nod}^-$.

We can write it more concisely as

$$\mathrm{d}_1^2(\ell,\ell') = d_1^2(\kappa_1) = \kappa_1^t \mathcal{A}_d \, \kappa_1 + B_d^t \kappa_1 + (d_{nod}^-)^2$$

where

$$\kappa_1 = (k_1', k_1) \; ; \qquad B_d = 2(B_1^-, B_2^-) \; ; \qquad \mathcal{A}_d = \begin{bmatrix} A_{11}^- & A_{12}^- \\ A_{21}^- & A_{22}^- \end{bmatrix} ;$$

with components

$$\begin{cases} B_1^- = 0 \\ B_2^- = d_{nod}^- \mathcal{F}_1 \end{cases} \qquad \begin{cases} A_{11}^- = a'^2 \\ A_{12}^- = A_{21}^- = \mathcal{G}_1 a' \cos I \\ A_{22}^- = [\mathcal{F}_1^2 + \mathcal{G}_1^2] \; . \end{cases}$$

The variable change to eliminate the linear terms in the quadratic form $d_1^2(\kappa_1)$ is defined by its inverse

$$\Xi_1^{-1} : \psi_1 \longrightarrow \kappa_1 = \mathcal{T}_1 \, \psi_1 + S_d$$

where $S_d = (S_1^-, S_2^-) \in \mathbb{R}^2$, $\psi_1 = (y_1', y_1) \in \mathbb{R}^2$ are the new variables and $\mathcal{T}_1$ is a 2×2 real-valued invertible matrix.

The equation to be solved in this case is

$$2 \, \mathcal{A}_d \, S_d + B_d = 0 \tag{4.23}$$

whose solutions are

$$S_1^- = \frac{B_2^- A_{12}^-}{det(\mathcal{A}_d)} = \frac{d_{nod}^- \mathcal{F}_1 \, \mathcal{G}_1 a' \cos I}{det(\mathcal{A}_d)} ;$$

$$S_2^- = -\frac{B_2^- A_{11}^-}{det(\mathcal{A}_d)} = -\frac{d_{nod}^- \mathcal{F}_1 a'^2}{det(\mathcal{A}_d)} \; .$$

We obtain

$$\mathcal{T}_1 = \frac{1}{\sqrt{det(\mathcal{A}_d)}} \begin{pmatrix} \rho_1 & -\sigma_1 \\ 0 & \tau_1 \end{pmatrix} = \begin{pmatrix} 1/\tau_1 & -\sigma_1/\tau_1\rho_1 \\ 0 & 1/\rho_1 \end{pmatrix} \tag{4.24}$$

where

$$\tau_1 = a' ; \qquad \sigma_1 = \mathcal{G}_1 \cos I ; \qquad \rho_1 = \frac{1}{a'} \sqrt{det(\mathcal{A}_d)} ;$$

and

$$det(\mathcal{A}_d) = a'^2 \left(\mathcal{F}_1^2 + \mathcal{G}_1^2 \sin^2 I \right) \; .$$

The minimal distance between the straight lines in this case is

$$d_{min}^- = |d_{nod}^-| \left[1 - \frac{a'^2 \mathcal{F}_1^2}{det(\mathcal{A}_1)} \right]^{1/2} = |d_{nod}^-| \left[\frac{a'^2 \mathcal{G}_1^2 \sin^2 I}{det(\mathcal{A}_d)} \right]^{1/2} \; .$$

Chapter 5

Hamilton's equations with $\overline{H}$

5.1 The derivatives of the averaged perturbing function $\overline{R}$

Kantorovich's method is used to describe the singularities of the derivatives of the averaged perturbing function with respect to Delaunay's variables appearing in equations (4.4).

Note that by the *chain rule* we can write

$$\frac{\partial \overline{R}}{\partial \mathcal{E}_{\mathcal{D}}} = \frac{\partial \overline{R}}{\partial \mathcal{E}_{\mathcal{K}}} \frac{\partial \mathcal{E}_{\mathcal{K}}}{\partial \mathcal{E}_{\mathcal{D}}}$$

where $\mathcal{E}_{\mathcal{K}} = \{e, I, \omega, \Omega\}$ is a subset of the Keplerian elements of the asteroid and

$$\frac{\partial \mathcal{E}_{\mathcal{K}}}{\partial \mathcal{E}_{\mathcal{D}}} = \begin{bmatrix} \mathcal{M} & O_2 \\ O_2 & I_2 \end{bmatrix},$$

in which I_2 and O_2 are the 2×2 identity and zero matrices, and

$$\mathcal{M} = -\frac{1}{k\sqrt{a}} \begin{bmatrix} \beta/e & 0 \\ -\mathrm{cotan} I/\beta & 1/(\beta \sin I) \end{bmatrix}.$$

Hence we can do the computations using the derivatives of $\overline{R}$ with respect to the Keplerian elements e, I, ω ($\overline{R}$ does not depend on Ω).

We shall not need to perform the splitting of Kantorovich's method to compute the derivative of $\overline{R}$ with respect to the inclination I; in fact we shall see that the derivative of $1/\mathfrak{D}$ with respect to I can be bounded by a function with a first order polar singularity in $\overline{u}, \overline{u}'$, so it is Lebesgue integrable over $\mathbb{T}^2$.

In the following we shall first prove that the derivatives of the remainder function $1/\mathrm{D} - 1/\mathrm{d}$ are always Lebesgue integrable over $\mathbb{T}^2$, and the integrals have finite values, even if the two orbits intersect each other. Hence the average of the remainder function is differentiable: indeed its derivatives can be computed by exchanging the position of the integral and differential operators as in (4.7). Then we shall see that, if there is an ascending node crossing, a discontinuous term appears in the derivatives of the average of $1/\mathrm{d}$, which is responsible of the discontinuity of the derivatives of $\overline{R}$. These derivatives admit two limit values at crossings (coming from the regions defined by $d^+_{nod} > 0$ and $d^+_{nod} < 0$).

As the properties we intend to prove are invariant by coordinate changes, we shall show them using the coordinates (u,u') instead of (ℓ,ℓ').

The derivatives of the remainder function $1/\mathfrak{D} - 1/\mathfrak{d}$.

Let us set $\upsilon = (u,u')$ and $\nu = (v,v') = (u-\overline{u}, u'-\overline{u}')$. We apply Taylor's formula with the integral remainder to the vector functions $P(u), P'(u')$:

$$\begin{cases} P(u) = P(\overline{u}) + P_u(\overline{u})\, v + \displaystyle\int_{\overline{u}}^{u} (u-s) P_{ss}(s)\, ds \\ P'(u') = P'(\overline{u}') + P'_{u'}(\overline{u}')\, v' + \displaystyle\int_{\overline{u}'}^{u'} (u'-t) P'_{tt}(t)\, dt\ . \end{cases}$$

The functions defining the straight lines $r(u) = \mathrm{r}(\ell(u))$ and $r'(u') = \mathrm{r}'(\ell'(u'))$ have the same Taylor's developments, up to the first order in $|\nu| = \sqrt{v^2 + v'^2}$, as $P(u)$ and $P'(u')$ respectively, so that we can write

$$\begin{cases} r(u) = P(\overline{u}) + P_u(\overline{u})\, v + \displaystyle\int_{\overline{u}}^{u} (u-s) r_{ss}(s)\, ds \\ r'(u') = P'(\overline{u}') + P'_{u'}(\overline{u}')\, v' + \displaystyle\int_{\overline{u}'}^{u'} (u'-t) r'_{tt}(t)\, dt\ . \end{cases}$$

We prove the following

THEOREM 14. *If there is an ascending node crossing at $(u,u') = (\overline{u},\overline{u}')$, the derivatives of the remainder function $1/\mathfrak{D} - 1/\mathfrak{d}$ with respect to e,ω can be bounded by functions having a first order polar singularity in $\overline{u},\overline{u}'$, so they are Lebesgue integrable over $\mathbb{T}^2$.*

Proof. We shall consider only the derivatives with respect to e: the proof for the other derivatives is similar. First we note that

$$\frac{\partial}{\partial e}\left[\frac{1}{\mathfrak{D}(\upsilon)}\right] = -\frac{1}{2\mathfrak{D}^3(\upsilon)}\frac{\partial}{\partial e}\left[\mathfrak{D}^2(\upsilon)\right]\ ; \qquad \frac{\partial}{\partial e}\left[\frac{1}{\mathfrak{d}(\upsilon)}\right] = -\frac{1}{2\mathfrak{d}^3(\upsilon)}\frac{\partial}{\partial e}\left[\mathfrak{d}^2(\upsilon)\right]\ .$$

Let us write $\langle\,,\rangle$ for the Euclidean scalar product. We have

$$\frac{\partial}{\partial e}\left[\mathfrak{D}^2(\upsilon)\right] = \mathfrak{D}^2_{e,0}(\upsilon) + \mathfrak{D}^2_{e,1}(\upsilon) + \mathfrak{D}^2_{e,2}(\upsilon) \tag{5.1}$$

where

$$\begin{aligned} \mathfrak{D}^2_{e,0}(\upsilon) &= 2\left\langle \frac{\partial}{\partial e}\left[P(u) - P'(u')\right], P(\overline{u}) - P'(\overline{u}') \right\rangle\ ; \\ \mathfrak{D}^2_{e,1}(\upsilon) &= 2\left\langle \frac{\partial}{\partial e}\left[P(u) - P'(u')\right], P_u(\overline{u})\, v - P'_{u'}(\overline{u}')\, v' \right\rangle\ ; \\ \mathfrak{D}^2_{e,2}(\upsilon) &= 2\left\langle \frac{\partial}{\partial e}\left[P(u) - P'(u')\right], \int_{\overline{u}}^{u} (u-s) P_{ss}(s)\, ds - \int_{\overline{u}'}^{u'} (u'-t) P'_{tt}(t)\, dt \right\rangle\ ; \end{aligned}$$

and

$$\frac{\partial}{\partial e}\left[\mathfrak{d}^2(\upsilon)\right] = \mathfrak{d}^2_{e,0}(\upsilon) + \mathfrak{d}^2_{e,1}(\upsilon) + \mathfrak{d}^2_{e,2}(\upsilon) \tag{5.2}$$

where

$$\begin{aligned}
\mathfrak{d}^2_{e,0(\upsilon)} &= 2\left\langle \frac{\partial}{\partial e}\left[r(u) - r'(u')\right], P(\overline{u}) - P'(\overline{u}') \right\rangle ;\\
\mathfrak{d}^2_{e,1}(\upsilon) &= 2\left\langle \frac{\partial}{\partial e}\left[r(u) - r'(u')\right], P_u(\overline{u})\, v - P'_{u'}(\overline{u}')\, v' \right\rangle ;\\
\mathfrak{d}^2_{e,2}(\upsilon) &= 2\left\langle \frac{\partial}{\partial e}\left[r(u) - r'(u')\right], \int_{\overline{u}}^{u} (u-s) r_{ss}(s)\, ds - \int_{\overline{u}'}^{u'} (u'-t) r'_{tt}(t)\, dt \right\rangle .
\end{aligned}$$

If we set the *crossing conditions* $P(\overline{u}) = P'(\overline{u}')$ we obtain

$$\mathfrak{D}^2_{e,0}(\upsilon) = \mathfrak{d}^2_{e,0}(\upsilon) = 0$$

and, in particular, the constant terms in Taylor's developments of $\partial\mathfrak{D}^2/\partial e$ and $\partial\mathfrak{d}^2/\partial e$ vanish.

The terms defined by $\mathfrak{D}^2_{e,2}$ and $\mathfrak{d}^2_{e,2}$ are at least infinitesimal of the second order with respect to $|\nu|$ as $\upsilon \to (\overline{u}, \overline{u}')$, so that the first order terms in $|\nu|$ at crossing can be given only by $\mathfrak{D}^2_{e,1}$ and $\mathfrak{d}^2_{e,1}$.

Using the theorems on the integrals depending on a parameter we obtain

$$\begin{aligned}
&\frac{\partial}{\partial e}\left[\int_{\overline{u}}^{u} (u-s) P_{ss}(s)\, ds - \int_{\overline{u}'}^{u'} (u'-t) P'_{tt}(t)\, dt\right] = \\
&\int_{\overline{u}}^{u} (u-s) \frac{\partial P_{ss}}{\partial e}(s)\, ds - \frac{\partial \overline{u}}{\partial e} P_{uu}(\overline{u})\, v - \int_{\overline{u}'}^{u'} (u'-t) \frac{\partial P'_{tt}}{\partial e}(t)\, dt + \frac{\partial \overline{u}'}{\partial e} P'_{u'u'}(\overline{u}')\, v' ;\\
&\frac{\partial}{\partial e}\left[\int_{\overline{u}}^{u} (u-s) r_{ss}(s)\, ds - \int_{\overline{u}'}^{u'} (u'-t) r'_{tt}(t)\, dt\right] = \\
&\int_{\overline{u}}^{u} (u-s) \frac{\partial r_{ss}}{\partial e}(s)\, ds - \frac{\partial \overline{u}}{\partial e} r_{uu}(\overline{u})\, v - \int_{\overline{u}'}^{u'} (u'-t) \frac{\partial r'_{tt}}{\partial e}(t)\, dt + \frac{\partial \overline{u}'}{\partial e} r'_{u'u'}(\overline{u}')\, v' ;
\end{aligned}$$

so that these two expressions are at least infinitesimal of the first order with respect to $|\nu|$. As these terms are multiplied by first order terms in the expressions of $\mathfrak{D}^2_{e,1}$ and $\mathfrak{d}^2_{e,1}$, they give rise to at least second order terms.

We can conclude that the first order terms in the expressions (5.1) and (5.2) are equal and they are given by

$$2\left\langle \frac{\partial}{\partial e}\left[P(\overline{u}) - P'(\overline{u}')\right] - \left[\frac{\partial \overline{u}}{\partial e} P_u(\overline{u}) - \frac{\partial \overline{u}'}{\partial e} P'_{u'}(\overline{u}')\right], P_u(\overline{u})\, v - P'_{u'}(\overline{u}')\, v' \right\rangle ;$$

therefore the asymptotic developments of the e-derivatives of $\mathfrak{D}^2(\upsilon)$ and $\mathfrak{d}^2(\upsilon)$ in a neighborhood of $\upsilon = (\overline{u}, \overline{u}')$ are

$$\frac{\partial}{\partial e}\left[\mathfrak{D}^2(\upsilon)\right] = \alpha v + \beta v' + \mathfrak{r}_{\mathfrak{D}}(\upsilon) ; \qquad \frac{\partial}{\partial e}\left[\mathfrak{d}^2(\upsilon)\right] = \alpha v + \beta v' + \mathfrak{r}_{\mathfrak{d}}(\upsilon) ;$$

where α, β are independent on u, u' and $\mathfrak{r}_{\mathfrak{D}}(\upsilon)$, $\mathfrak{r}_{\mathfrak{d}}(\upsilon)$ are infinitesimal of the second order with respect to $|\nu|$ as $\upsilon \to (\overline{u}, \overline{u}')$.

Using the decomposition

$$\left[\frac{1}{\mathfrak{D}^3} - \frac{1}{\mathfrak{d}^3}\right] = \left[\frac{1}{\mathfrak{D}} - \frac{1}{\mathfrak{d}}\right]\left[\frac{1}{\mathfrak{D}^2} + \frac{1}{\mathfrak{D}\mathfrak{d}} + \frac{1}{\mathfrak{d}^2}\right] ,$$

the boundedness of the remainder function $1/\mathfrak{D} - 1/\mathfrak{d}$ and lemma 2 (that also hold in the (u, u') coordinates), we conclude that there exist two constants $L_1, L_2 > 0$ such that

$$\left|\frac{\partial}{\partial e}\left[\frac{1}{\mathfrak{D}(\upsilon)}\right] - \frac{\partial}{\partial e}\left[\frac{1}{\mathfrak{d}(\upsilon)}\right]\right| = \frac{1}{2}\left|\left\{\left[\frac{1}{\mathfrak{D}^3(\upsilon)} - \frac{1}{\mathfrak{d}^3(\upsilon)}\right](\alpha v + \beta v') + \right.\right.$$

$$\left.\left. + \frac{1}{\mathfrak{D}^3(\upsilon)}\mathfrak{r}_{\mathfrak{D}}(\upsilon) - \frac{1}{\mathfrak{d}^3(\upsilon)}\mathfrak{r}_{\mathfrak{d}}(\upsilon)\right\}\right| \leq L_1\frac{1}{|\nu|} + L_2$$

in a neighborhood of $\upsilon = (\overline{u}, \overline{u}')$ and the theorem is proven.

□

Singularities of the $\{e, \omega\}$-derivatives of the average of $1/\mathrm{d}$.

As $det(\mathcal{A}) > 0$ and $(\overline{\ell}, 0)$ is in the interior part of $\mathbb{T}^2$, we have $d^2_{min} + r^2_j(\theta) > 0$ for each $\theta \in [\theta_j, \theta_{j+1}]$ and for each $j = 1 \ldots 4$. Then we can use again the theorem of differentiation under the integral sign and compute, for instance, the derivative of the average of $1/\mathrm{d}$ with respect to e as

$$\frac{\partial}{\partial e}\int_{\mathbb{T}^2}\frac{1}{\mathrm{d}}\,d\ell\, d\ell' = \frac{\partial}{\partial e}\left[\frac{1}{\sqrt{det(\mathcal{A})}}\right]\cdot\left\{\sum_{j=1}^{4}\int_{\theta_j}^{\theta_{j+1}}\sqrt{(d^+_{min})^2 + r^2_j(\theta)}\,d\theta - 2\pi d^+_{min}\right\} +$$

$$+\left[\frac{1}{\sqrt{det(\mathcal{A})}}\right]\cdot\left\{\frac{1}{2}\sum_{j=1}^{4}\int_{\theta_j}^{\theta_{j+1}}\frac{\frac{\partial}{\partial e}[(d^+_{min})^2 + r^2_j(\theta)]}{\sqrt{(d^+_{min})^2 + r^2_j(\theta)}}\,d\theta - 2\pi\frac{\partial}{\partial e}d^+_{min}\right\} . \tag{5.3}$$

We have similar formulas for the derivatives with respect to ω, obtained simply by substitution of the partial derivative operators.

The discontinuities present in the terms

$$\frac{\partial}{\partial e}d^+_{min}; \qquad \frac{\partial}{\partial \omega}d^+_{min};$$

are responsible of the discontinuities in the derivatives of the averaged perturbing function that produce a sort of *crests* in the surfaces representing this function (see Figure 5.1) and cause the loss of regularity in its level lines, where the weak averaged solutions lie.

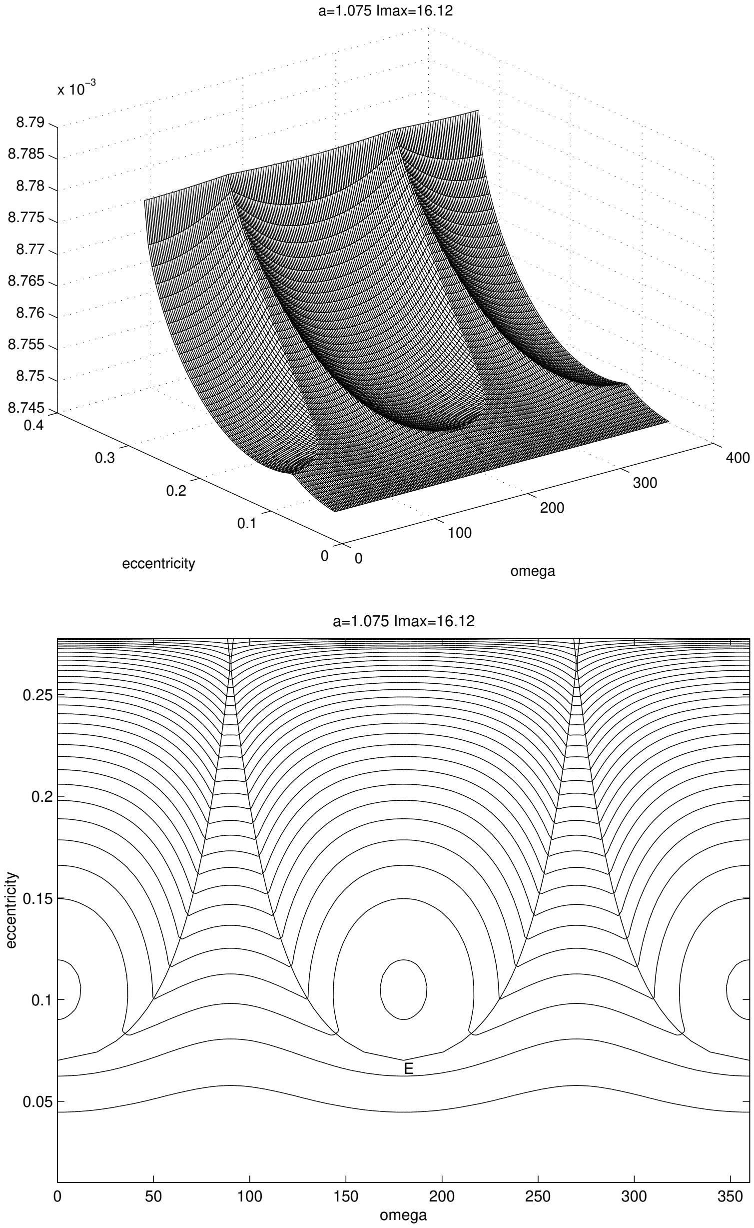

Figure 5.1: We draw the graphic of the averaged perturbing function (top) and its level lines (bottom) in the plane (ω, e) for the Near Earth Asteroid 2000 CO_{101} (ω is in degrees in this figure). The loss of regularity at the node crossing lines with the Earth is particularly evident for this object.

5.1.1 Singularities of the I-derivative of $\overline{R}$

We observe that we do not need to perform the splitting of Kantorovich's method to compute the derivative with respect to I of the integral average of $1/\mathfrak{D}$, in fact we have the following

PROPOSITION 7. *If there is an ascending node crossing, the derivative of* $1/\mathfrak{D}$ *with respect to the mutual inclination I can be bounded by a function with a first order polar singularity in* $\overline{u},\overline{u}'$*, so it is Lebesgue integrable over* $\mathbb{T}^2$.

Proof. We have

$$\frac{\partial}{\partial I}\left[\frac{1}{\mathfrak{D}(\upsilon)}\right] = -\frac{1}{2\mathfrak{D}^3(\upsilon)}\frac{\partial}{\partial I}\left[\mathfrak{D}^2(\upsilon)\right],$$

so we need to prove that $\partial\mathfrak{D}^2/\partial I$ is an infinitesimal of the second order with respect to $|\nu|$ as $\upsilon \to (\overline{u},\overline{u}')$. For this purpose we only need to check the vanishing of the term

$$2\left\langle \frac{\partial}{\partial I}\left[P(\overline{u}) - P'(\overline{u}')\right] - \left[\frac{\partial\overline{u}}{\partial I}P_u(\overline{u}) - \frac{\partial\overline{u}'}{\partial I}P'_{u'}(\overline{u}')\right], P_u(\overline{u})\,\nu - P'_{u'}(\overline{u}')\,\nu'\right\rangle \tag{5.4}$$

that formally represents the first order terms in the derivative of $\mathfrak{D}^2(\upsilon)$ with respect to I. as we can see from similar computations in Theorem 14. The expression in (5.4) vanishes because $P(\overline{u}), P'(\overline{u}'), \overline{u}$ and $\overline{u}'$ do not depend on I, in fact

$$\cos\overline{u}\sin\omega + \beta\sin\overline{u}\cos\omega = 0\ .$$

□

5.1.2 The averaged equations and their discontinuities

In this section we describe in details the averaged equations of motion: we also put in evidence the discontinuities of these equations giving analytical formulas for them. This is done by means of Kantorovich's decomposition, that allows us to give an *operative definition* of the weak averaged solutions.

REMARK 14. It is possible that for very peculiar choices of the initial conditions the definition of a weak averaged solution is not unique but there are two ways of continuing the solution. This happens when the level line of the averaged Hamiltonian, that constrains the orbit in the phase space (ω, e), reaches a node crossing line being tangent to it. This is possible when the crossing point corresponds to a local minimum of the averaged perturbing function restricted to the node crossing line (see Chapter 6).

Description of the differential equations.

Let us consider the perturbing function

$$R = k^2\mu\frac{1}{\mathrm{D}} \qquad\qquad (\mu = m'/m_\odot)$$

as a function of the mean anomalies ℓ, ℓ'. The averaged equations that we want to solve, in terms of the orbital elements, are

$$
\begin{aligned}
\dot{\omega} &= -\frac{\partial \overline{R}}{\partial G} = -\frac{\partial \overline{R}}{\partial e}\Big[\frac{-\beta}{k\sqrt{a}e}\Big] + \frac{\partial \overline{R}}{\partial I}\Big[\frac{cotg I}{k\sqrt{a}\beta}\Big] = \\
&= k^2\mu\Big\{\Big[\frac{\beta}{k\sqrt{a}e}\Big]\Big[\frac{\partial}{\partial e}\int_T \frac{1}{\mathrm{d}}\, d\ell\, d\ell' + \int_T \frac{\partial}{\partial e}\Big(\frac{1}{\mathrm{D}} - \frac{1}{\mathrm{d}}\Big)\, d\ell\, d\ell'\Big] - \\
&\quad -\Big[\frac{cotg I}{k\sqrt{a}\beta}\Big]\Big[\int_T \frac{\partial}{\partial I}\Big(\frac{1}{\mathrm{D}}\Big)\, d\ell\, d\ell'\Big]\Big\}\ ; \\
\dot{G} &= \frac{\partial \overline{R}}{\partial \omega} = \frac{\partial}{\partial \omega}\int_T \frac{1}{\mathrm{d}}\, dl\, dl' + \int_T \frac{\partial}{\partial \omega}\Big(\frac{1}{\mathrm{D}} - \frac{1}{\mathrm{d}}\Big)\, d\ell\, d\ell'\ ; \\
\dot{\Omega} &= -\frac{\partial \overline{R}}{\partial Z} = -\frac{\partial \overline{R}}{\partial I}\Big[\frac{-1}{k\sqrt{a}\beta\sin I}\Big] = \Big[\frac{1}{k\sqrt{a}\beta\sin I}\Big]\int_T \frac{\partial}{\partial I}\Big(\frac{1}{\mathrm{D}}\Big)\, d\ell\, d\ell'\ .
\end{aligned}
$$

Discontinuities of the equations.

We present the formulas for the discontinuities of the derivatives of $\overline{R}$.

Let

$$
\frac{\partial^+}{\partial e}\ ;\ \frac{\partial^+}{\partial \omega}\ ; \qquad \text{and} \qquad \frac{\partial^-}{\partial e}\ ;\ \frac{\partial^-}{\partial \omega}\ ;
$$

be the partial derivative operators applied in the regions of the space where $d^+_{nod} > 0$ and $d^+_{nod} < 0$ respectively, that is the partial derivatives of the restriction of a function to these domains.

We define the operator 'Diff' to describe the differences in the right hand sides of equations (4.3) at $d^+_{nod} = 0$ when we pass from a region where $d^+_{nod} > 0$ to a region where $d^+_{nod} < 0$; this definition is given taking into account the existence of a continuous extension of the derivatives to the boundaries of both the sets $\{d^+_{nod} > 0\} \cap W$ and $\{d^+_{nod} < 0\} \cap W$, where W is the Kozai domain. We have

$$
\begin{aligned}
\mathrm{Diff}\left(\frac{\partial \overline{R}}{\partial G}\right) &= \frac{k\mu(1-e^2)}{2\pi\sqrt{det(\mathcal{A})}\cdot e\sqrt{a}} \cdot \Big[1 - \frac{a^2 a'^2 e^2 \sin^2\omega}{(1-e^2)det(\mathcal{A})}\Big]^{\frac{1}{2}} \cdot \mathrm{Diff}\left(\frac{\partial |d^+_{nod}|}{\partial e}\right)\ ; \\
\mathrm{Diff}\left(\frac{\partial \overline{R}}{\partial \omega}\right) &= -\frac{k^2\mu}{2\pi\sqrt{det(\mathcal{A})}} \cdot \Big[1 - \frac{a^2 a'^2 e^2 \sin^2\omega}{(1-e^2)det(\mathcal{A})}\Big]^{\frac{1}{2}} \cdot \mathrm{Diff}\left(\frac{\partial |d^+_{nod}|}{\partial \omega}\right)\ ; \\
\mathrm{Diff}\left(\frac{\partial \overline{R}}{\partial Z}\right) &= 0\,;
\end{aligned}
$$

where

$$
\begin{aligned}
\mathrm{Diff}\left(\frac{\partial}{\partial e}|d^+_{nod}|\right) &= \Big[\frac{\partial^+ |d^+_{nod}|}{\partial e} - \frac{\partial^- |d^+_{nod}|}{\partial e}\Big] = \frac{2a(\cos\omega(1+e^2)+2e)}{(1+e\cos\omega)^2}\,; \\
\mathrm{Diff}\Big(\frac{\partial}{\partial \omega}|d^+_{nod}|\Big) &= \Big[\frac{\partial^+ |d^+_{nod}|}{\partial \omega} - \frac{\partial^- |d^+_{nod}|}{\partial \omega}\Big] = -\frac{2a(1-e^2)e\sin\omega}{(1+e\cos\omega)^2}\ .
\end{aligned}
$$

REMARK 15. In the last formula the quantities

$$\frac{\partial^+|d^+_{nod}|}{\partial e}\,; \qquad \frac{\partial^-|d^+_{nod}|}{\partial e}\,; \qquad \frac{\partial^+|d^+_{nod}|}{\partial \omega}\,; \qquad \frac{\partial^-|d^+_{nod}|}{\partial \omega}\,;$$

have to be considered as the extensions of these functions to the points of the common boundary $d^+_{nod} = 0$.

5.2 Different kind of node crossings

In this section we shall briefly discuss the remaining two cases: *crossings at the descending node* and *double crossings*.

5.2.1 Descending node crossings

Let us assume that a descending node crossing occurs between the orbits: we can define weak averaged solutions of equations (3.23) following a procedure similar to the one described in the previous sections.

We shall use the inverse of the Wetherill function $1/\mathrm{d}_1$ to extract the principal part from the direct term of the perturbing function. We observe that the point $(\overline{\ell}_1, \pi)$ is on the boundary of the torus $\mathbb{T}^2 = \{(\ell,\ell') : -\pi \le \ell \le \pi, -\pi \le \ell' \le \pi\}$: we use the periodicity of the function D and consider the shifted domain

$$\tau_{\pi,0}\left(\mathbb{T}^2\right) = \{(\ell,\ell') : -\pi \le \ell \le \pi, 0 \le \ell' \le 2\pi\}\ .$$

so that the crossing value $(\overline{\ell}_1, \pi)$ is generally an internal point.

In computing the derivatives of $\overline{R}$ with respect to the variables $\mathcal{E}_{\mathcal{D}}$, for instance the G-derivative, we can use the decomposition

$$\frac{(2\pi)^2}{\mu k^2}\frac{\partial}{\partial G}\overline{R} = \int_{\tau_{\pi,0}\left(\mathbb{T}^2\right)} \frac{\partial}{\partial G}\left[\frac{1}{\mathrm{D}} - \frac{1}{\mathrm{d}_1}\right] d\ell\, d\ell' + \frac{\partial}{\partial G}\int_{\tau_{\pi,0}\left(\mathbb{T}^2\right)} \frac{1}{\mathrm{d}_1}\, d\ell\, d\ell'\ . \qquad (5.5)$$

Discontinuities in the derivatives: in case of crossing at the descending node, we have the following formulas:

$$det(\mathcal{A}_d) = \frac{a^2 a'^2}{(1-e^2)} \cdot (\sin^2 I(1-2e\cos\omega + e^2) + e^2\cos^2 I \sin^2\omega)\,;$$

$$\begin{aligned}
\mathrm{Diff}_1\left(\frac{\partial}{\partial e}|d^-_{nod}|\right) &= \frac{\partial^+_1|d^-_{nod}|}{\partial e} - \frac{\partial^-_1|d^-_{nod}|}{\partial e} = -\frac{2a(\cos\omega(1+e^2)-2e)}{(1-e\cos\omega)^2}\,;\\
\mathrm{Diff}_1\left(\frac{\partial}{\partial \omega}|d^-_{nod}|\right) &= \frac{\partial^+_1|d^-_{nod}|}{\partial \omega} - \frac{\partial^-_1|d^-_{nod}|}{\partial \omega} = \frac{2a(1-e^2)e\sin\omega}{(1-e\cos\omega)^2}\,;
\end{aligned}$$

where

$$\frac{\partial_1^+}{\partial e};\ \frac{\partial_1^+}{\partial \omega}; \qquad \text{and} \qquad \frac{\partial_1^-}{\partial e};\ \frac{\partial_1^-}{\partial \omega};$$

are the partial derivative operators applied to the restriction to the regions of the space where $d_{nod}^- > 0$ and $d_{nod}^- < 0$ respectively, and the operator 'Diff$_1$' describes the differences in the right hand sides of equations (4.3) at $d_{nod}^- = 0$, when we pass from a region where $d_{nod}^- > 0$ to a region where $d_{nod}^- < 0$.

5.2.2 Double crossings

In the case of double crossings we use both the inverse of the Wetherill functions $1/\mathrm{d}$ and $1/\mathrm{d}_1$ to extract the principal part from the direct term of the perturbing function. We observe that the point $(\overline{\ell}, 0)$ is on the boundary of $\tau_{\pi,0}\left(\mathbb{T}^2\right)$ and the point $(\overline{\ell}_1, \pi)$ is on the boundary of $\mathbb{T}^2$; we use again the periodicity of D and consider the shifted domain

$$\tau_{\pi/2,0}\left(\mathbb{T}^2\right) = \{(\ell, \ell') : -\pi \le \ell \le \pi, -\pi/2 \le \ell' \le 3\pi/2\},$$

so that the crossing values $(\overline{\ell}, 0)$ and $(\overline{\ell}_1, \pi)$ are generally internal points.

In computing the derivatives of $\overline{R}$ with respect to the variables $\mathcal{E}_{\mathcal{D}}$, for instance the G-derivative, we can use the decomposition

$$\frac{(2\pi)^2}{\mu k^2}\frac{\partial}{\partial G}\overline{R} = \int_{\tau_{\pi/2,0}\left(\mathbb{T}^2\right)} \frac{\partial}{\partial G}\left[\frac{1}{\mathrm{D}} - \frac{1}{\mathrm{d}} - \frac{1}{\mathrm{d}_1}\right] d\ell\, d\ell' + \frac{\partial}{\partial G}\int_{\tau_{\pi/2,0}\left(\mathbb{T}^2\right)} \frac{1}{\mathrm{d}}\, d\ell\, d\ell' +$$
$$+\frac{\partial}{\partial G}\int_{\tau_{\pi/2,0}\left(\mathbb{T}^2\right)} \frac{1}{\mathrm{d}_1}\, d\ell\, d\ell'\,.$$

5.3 Derivatives of $\overline{R}$ along the node crossing lines

For each point $(\overline{\omega}, \overline{e})$ on a node crossing line in the phase space we consider the unit vector $\tau_{\overline{\omega},\overline{e}}$, tangent to the line at that point and we study the regularity properties of the averaged Hamiltonian $\overline{H}$ restricted to the node crossing line. It comes out that the Hamiltonian is derivable in each point $(\overline{\omega}, \overline{e})$ of this curve along the direction of $\tau_{\overline{\omega},\overline{e}}$, the tangent vector to the curve in $(\overline{\omega}, \overline{e})$.

The following result is easily shown:

Lemma 3. *Let $F(x,y) \in C^\infty(\Omega;\mathbb{R})$ where $\Omega \in \mathbb{R}^2$ is an open domain. Let us assume that $\nabla F(x,y) \neq 0$ for each $(x,y) \in \Omega$ and let us consider the curve γ defined by the equation $F(x,y) = 0$. If for each in $(\overline{x}, \overline{y}) \in \Omega$ we call $\tau_{\overline{x},\overline{y}}$ the unit vector tangent to γ in $(\overline{x}, \overline{y})$ we have that the function $|F(x,y)|$ is derivable in $(\overline{x}, \overline{y})$ along the direction defined by $\tau_{\overline{x},\overline{y}}$, and we have*

$$\frac{\partial |F|}{\partial \tau_{\overline{x},\overline{y}}}(\overline{x}, \overline{y}) = 0\,.$$

Proof. Take $(\overline{x},\overline{y}) \in \Omega$ and assume that $\partial_x F(\overline{x},\overline{y}) \neq 0$, where ∂_x is the partial derivative with respect to x. We can write a vector tangent to γ in $(\overline{x},\overline{y})$ as

$$\tau_{\overline{x},\overline{y}} = (1, \beta(\overline{x},\overline{y})) \qquad \text{with} \qquad \beta(\overline{x},\overline{y}) = -\frac{\partial_x F(\overline{x},\overline{y})}{\partial_y F(\overline{x},\overline{y})} .$$

The derivative of $|F|$ in $(\overline{x},\overline{y})$, along the direction of $\tau_{\overline{x},\overline{y}}$, is defined by

$$\frac{\partial |F|}{\partial \tau_{\overline{x},\overline{y}}}(\overline{x},\overline{y}) = \lim_{h\to 0}\left\{\frac{|F(\overline{x}+h,\overline{y}+h\beta(\overline{x},\overline{y}))| - |F(\overline{x},\overline{y})|}{h}\right\} = \lim_{h\to 0}\left\{\frac{|F(\overline{x}+h,\overline{y}+h\beta(\overline{x},\overline{y}))|}{h}\right\} .$$

Furthermore we have

$$\begin{aligned} &F(\overline{x}+h,\overline{y}+h\beta(\overline{x},\overline{y})) = F(\overline{x},\overline{y}) + h\partial_x F(\overline{x},\overline{y}) + h\beta(\overline{x},\overline{y})\partial_y F(\overline{x},\overline{y}) + o(h) = \\ &= h\left\{\partial_x F(\overline{x},\overline{y}) + \beta(\overline{x},\overline{y})\partial_y F(\overline{x},\overline{y})\right\} + o(h) = \\ &= h\left\{\partial_x F(\overline{x},\overline{y}) - \frac{\partial_x F(\overline{x},\overline{y})}{\partial_y F(\overline{x},\overline{y})}\partial_y F(\overline{x},\overline{y})\right\} + o(h) = o(h) . \end{aligned}$$

The proof under the assumption that $\partial_y F(\overline{x},\overline{y}) \neq 0$ is quite similar.

□

Let us consider the smooth function d^+_{nod}, whose zeros define the ascending node crossing line. We observe that the gradient of d^+_{nod} is never zero in the interior part of the Kozai domain: in fact

$$\begin{aligned} \frac{\partial}{\partial e} d^+_{nod} &= -\frac{a(2e + (1+e^2)\cos\omega)}{(1+e\cos\omega)^2} ; \\ \frac{\partial}{\partial \omega} d^+_{nod} &= \frac{ae(1-e^2)\sin\omega}{(1+e\cos\omega)^2} ; \end{aligned}$$

so that if $\sin\omega = 0$ we have either

$$2e + (1+e^2)\cos\omega = (1+e)^2 > 0$$

or

$$2e + (1+e^2)\cos\omega = -(1-e)^2 < 0 .$$

Let us fix a point $(\overline{\omega},\overline{e})$ on the node crossing line. We can prove, following the steps of Section 5.1, that the only term that can be responsible of a discontinuity at $(\overline{\omega},\overline{e})$ in the derivatives of the averaged perturbing function along the direction $\tau_{\overline{\omega},\overline{e}}$ tangent to $(\overline{\omega},\overline{e})$ is

$$\frac{\partial |d^+_{nod}|}{\partial \tau_{\overline{\omega},\overline{e}}} ;$$

by the previous lemma this derivative exists in $(\overline{\omega},\overline{e})$ and it is zero.

The proof for the derivatives of $\overline{R}$ along the directions tangent to the descending node lines is quite similar and will be omitted.

Chapter 6

The stable Kozai state

6.1 Level curves of $\overline{H}$

In this chapter we give an accurate description of all the possible trajectories of the weak averaged solutions in the circular coplanar case. We shall also prove a Theorem that assures the existence of a point of the Kozai domain W (see Section 4.2) in a neighborhood of which all the trajectories avoid crossings with all the planets in the model. This occurs for any finite number of perturbing planets with any masses and for each value of the integrals a, Z. These results have been presented in [29], and they are relevant, for instance, in computing the impact probability of the asteroids with the Solar System planets.

To understand the geometry of the trajectories of the weak averaged solutions we look at the changes occurring for a fixed value of a, as the other integral Z, and therefore I_{max} and e_{max}, vary.

In the reduced phase space (ω, e) we show three different groups of figures, that correspond to three different values of the semimajor axis a, showing the level lines of the averaged Hamiltonian $\overline{H}$ (see Figures 6.1, 6.2, 6.3). In these plots the variable ω takes its values in $[0, \pi/2]$; the $2\pi-$periodicity of $\overline{H}$ with respect to ω and the invariance under the symmetries

$$
\begin{aligned}
\omega &\longrightarrow -\omega \\
\omega &\longrightarrow \pi - \omega \\
\omega &\longrightarrow \pi + \omega
\end{aligned}
$$

allows us to know the values of $\overline{H}$ in the whole Kozai domain.

The example in Figure 6.1 is representative of an outer Main Belt Asteroid, with $a =$ 3.41 AU (like the asteroid *(1373) Cincinnati*, the example originally used by Kozai in [39]). For low I_{max}, the value of e_{max} is so low that close approaches to Jupiter can not occur, and the level lines of $\overline{H}$ correspond to almost constant e. As I_{max} increases, and therefore also e_{max} increases, the level lines bend (top left) in such a way that e is maximum for $\omega = \pi/2$ and minimum for $\omega = 0$: this dynamical mechanism is called $e - \omega$ *coupling*. For a critical value of e_{max}, a bifurcation occurs at the point $e = 0$ (top

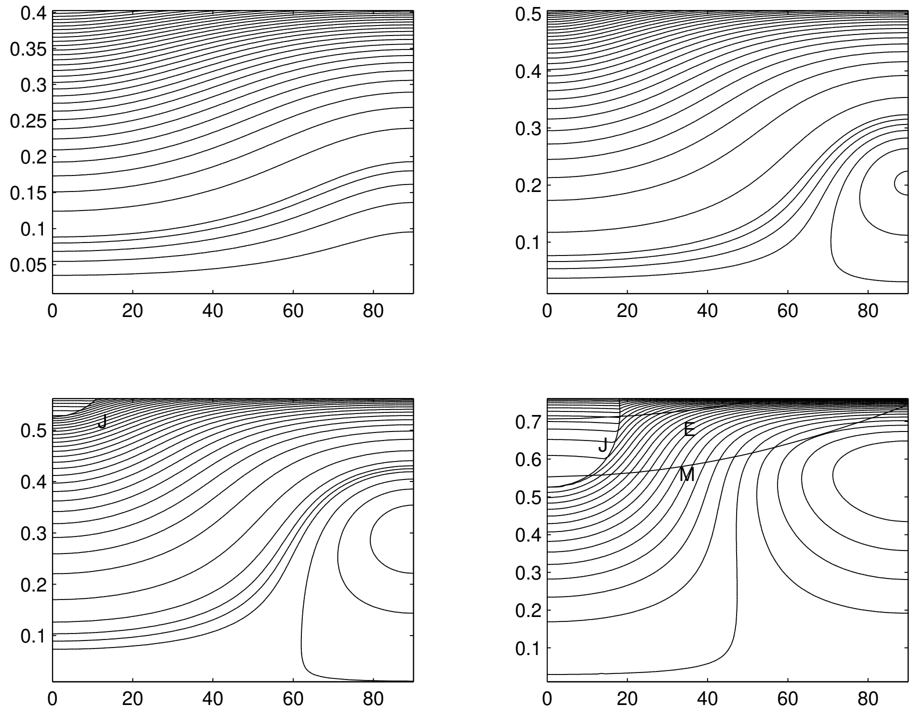

Figure 6.1: Level lines of the averaged Hamiltonian $\overline{H}$, in the (ω, e) plane, for $a = 3.41$ AU (the angle ω is in degrees). Node crossing lines with the Earth (E), Mars (M) and Jupiter (J) are plotted when they appear inside the Kozai domain. Top left: for $I_{max} = 25°$ there is a significant $e - \omega$ coupling. Top right: for $I_{max} = 31.°33$ the Kozai bifurcation has occurred, and libration of ω around 90° is possible. Bottom left: for $I_{max} = 35°$ the libration region is larger, and e_{max} is such that node crossings with Jupiter can occur for $\omega \simeq 0°$. Bottom right: for $I_{max} = 50°$ the libration region extends to large eccentricities, and is intersected by the node crossing line with Mars. Node crossings with the Earth are also possible for $e > 0.7$, but not in the libration region.

right): a maximum of the function $\overline{H}$ splits into two symmetric maxima and a saddle; the maxima have to bifurcate at least in pairs, because of the symmetry.

For $a(1 + e_{max}) > a'$ (bottom left) the node crossing lines of an exterior planet (in this case, Jupiter) appear along the $\omega = 0$ axis. As e_{max} further increases, the *Kozai separatrix* (see [39]), that is the level curve through the saddle, increases in size and two large regions of *libration* appear, with ω permanently oscillating around $\pi/2$ (bottom left). For large e, the node crossing line with Jupiter *pushes* the Kozai separatrix, squeezing the libration regions to higher eccentricity (bottom right); the libration regions are intersected by the node crossing lines with Mars, but not with the Earth.

The example in Figure 6.2 refers to an Earth crossing orbit, with semimajor axis $a = 1.1$ AU. For comparatively low values of e_{max} (top left) the node crossing line with the Earth is already present in the Kozai domain, but the argument of perihelion ω circulates . For larger values of e_{max}, allowing one nodal point well inside 1 AU, a Kozai libration appears around $\omega = 0$ (top right). With the further increasing of e_{max}, also the

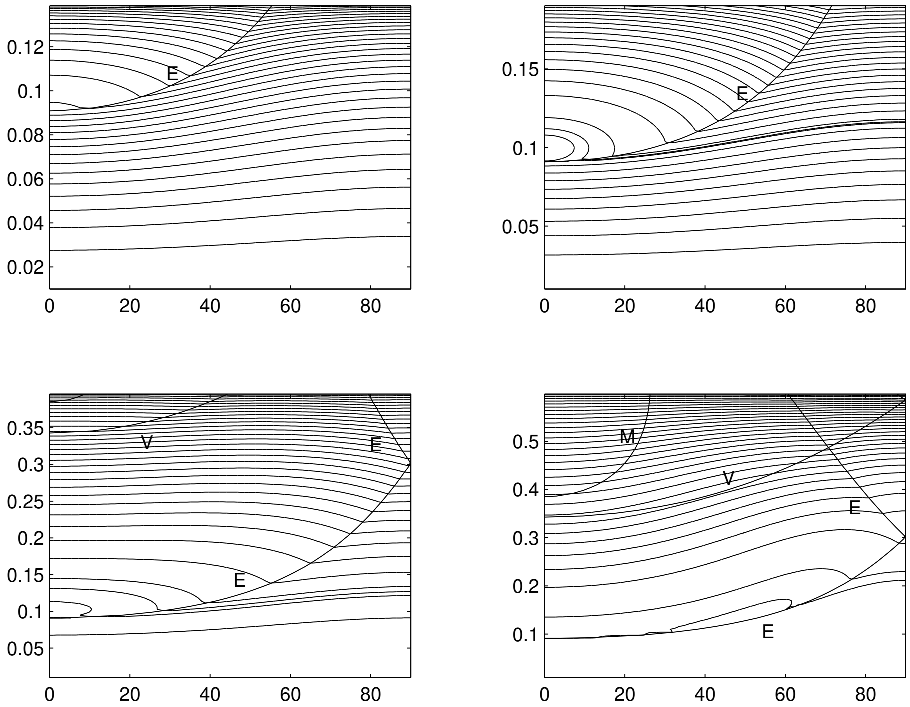

Figure 6.2: Level lines of $\overline{H}$ for $a = 1.1$ AU; node crossing lines with Venus, the Earth, and Mars are also plotted, if present. For $I_{max} = 8.^\circ$ node crossings with the Earth can occur (top left), but the only maximum is for $e = 0$, and there is no libration. For $I_{max} = 11.^\circ 5$ there is a symmetric libration around $\omega = 0^\circ$ (top right). Note that the level lines are not uniformly spaced, to enhance the libration region. For $I = 23.^\circ 5$ there is also the line of node crossing with Venus, and the size of the libration region decreases (bottom left); the node crossing line with Mars also appears in the top left corner of the figure. For $I_{max} = 37.^\circ$ the symmetric libration region has been replaced by much smaller asymmetric libration regions (bottom right); the corresponding maximum is very shallow, to the point that enhancing the libration region by denser level lines results in graphic difficulties.

node crossing line with Venus appears (bottom left), and the size of the Kozai libration region decreases. For large e_{max} the symmetric Kozai libration bifurcates into asymmetric librations (bottom right); the stable libration regions are very small, and correspond to very shallow maxima; level curves for very close values of $\overline{H}$ are required, and a finer grid than the one used in this figure would be required to give them a smooth appearance. We observe that circulation of ω without node crossings is always possible for low values of e.

By comparing the two examples above, we can see that the Kozai libration, around either $\omega = \pi/2$ or $\omega = 0$, prevents the secular evolution of the asteroid from node crossings and close approaches. When a close approach can occur near the perihelion (resp. aphelion), the argument of perihelion ω oscillates around values displacing as much as possible the perihelion (resp. aphelion) from the plane of the perturbing planets; this results in a protection mechanism preventing approaches to Jupiter for the outer Main Belt

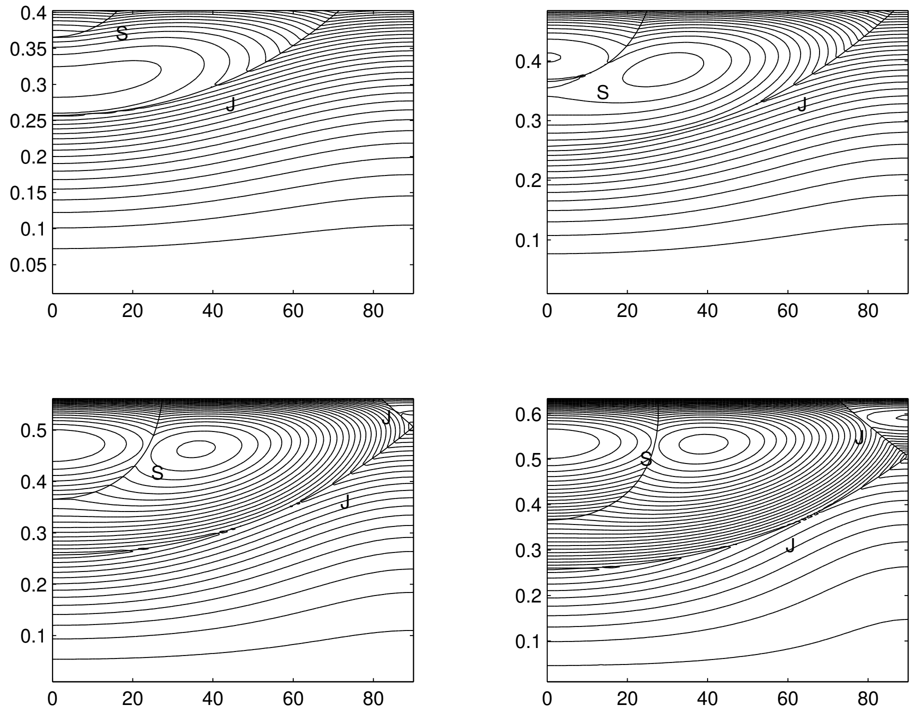

Figure 6.3: Level lines of $\overline{H}$ for $a = 7$ AU; node crossing lines with both Jupiter and Saturn are also plotted. Top left: for $I_{max} = 25°$ a libration region appears around $\omega = 0°$, between the node crossing with Jupiter (below) and with Saturn (above). Top right: for $I_{max} = 30°$ there is an asymmetric libration region, and a symmetric one for e above the Saturn crossing line. Bottom left: for $I_{max} = 35°$ a minimum of $\overline{H}$ occurs at the point where the two node crossing lines with Jupiter intersect; it is surrounded by a web-like libration region, where the average solutions undergo four node crossings per period of libration of ω. Bottom right: for $I_{max} = 40°$ a new libration region around $\omega = 90°$ becomes evident for large eccentricity.

Asteroids such as *(1373) Cincinnati* (see [39]), and close approaches to the Earth for asteroids such as *(3040) Kozai* (see [50]). When close approaches could occur at distances from the Sun near the value of a, then the perihelion and the aphelion are kept on the reference plane, so that the intermediate values of the heliocentric distance take place as much as possible out of the plane (see [49]).

The third example refers to *Centaur* type orbits, with semimajor axis $a = 7$ AU, that is an intermediate value between those of Jupiter and Saturn (Figure 6.3).

For moderate values of I_{max} the node crossing line with Jupiter and Saturn appears (top left), and a bifurcation takes place, generating a couple of symmetric maxima along $\omega = 0$; they correspond to orbits with one nodal point inside the orbit of Saturn, but outside the orbit of Jupiter. For larger values of I_{max}, the line of node crossing with Saturn approaches to the libration region (top right), and two more bifurcations take place: the two libration centers with $\omega = 0$ split into four asymmetric libration centers with one nodal point inside Jupiter and the other one between Jupiter and Saturn; a couple of symmetric libration centers appear, with $\omega = 0, \pi$ and a nodal point beyond Saturn. For larger I_{max} a double crossing point appears on $\omega = \pi/2$ (bottom left); this is a non-smooth minimum point of $\overline{H}$, surrounded by a web-like region, such that the weak averaged solutions undergo four node crossings per period of libration of ω. For even larger values of I_{max} (bottom right) a new libration region, protected from node crossings, appears clearly beyond this region, that is with both nodal points inside the orbit of Jupiter.

To conclude this heuristic description we summarize in the following scheme all the possibilities for the level lines of the averaged Hamiltonian that we have detected; they represent the possible trajectories of the weak averaged solutions:

1. circulation of ω in the (ω, e) plane;
2. couples of symmetric librations with two libration centers, either with centers in $\omega = 0, \pi$ or with centers in $\omega = \pi/2, 3\pi/2$;
3. quartets of asymmetric librations around $\omega = \alpha, -\alpha, \pi + \alpha, \pi - \alpha$ for some α.

These three different kinds of behavior can be either *protective* or *destructive*, as they can force the level lines of the averaged Hamiltonian to avoid or to go across the node crossing lines with the planets.

REMARK 16. The circulation of ω also corresponds to a libration with center in the origin if we consider the coordinate system (ξ, η), described in Section 4.2.

6.2 Existence of a stable Kozai state

In this section we prove the existence of a stable state, around which stable oscillations avoiding crossings are possible. Let us give the following

DEFINITION 18. *Given a value of the semimajor axis a and a value of the third component of the angular momentum $Z = k\sqrt{a(1-e^2)}$, that both are integrals of the averaged problem, we call* stable Kozai state *a stable equilibrium point of the dynamical system defined by the averaged equations (4.4) such that it is situated neither on the node crossing lines with the planets considered in the model, nor on the boundary of the Kozai domain $e = e_{max}$..*

We shall prove that each local maximum of the averaged Hamiltonian $\overline{H} = \overline{H}(\omega, e)$ is a stable Kozai state. Note that the existence of such states has an immediate influence on the collision probability computation, see [7], [40].

THEOREM 15. *(see [29]) At least one stable Kozai state occurs for all values of the integrals a, Z, for an arbitrary number of perturbing planets, and for all values of the parameters (semimajor axes and masses of the planets).*

Proof. We shall prove that in a neighborhood of the local maximum points of the averaged Hamiltonian $\overline{H}$ there are always stable Kozai librations in the phase space (ξ, η), such that the trajectories of the weak averaged solutions never cross any node crossing line.

Let us fix a value for the integrals a, Z and for the semimajor axes and masses of the perturbing planets in our Solar System model. The averaged Hamiltonian $\overline{H}$, regarded as a function of (ξ,η), is continuous on the compact Kozai domain W, thus there exists a value where the function attains its maximum and, in particular, there exists at least one local maximum point for $\overline{H}$.

We recall that $\overline{H} = H_0 - \overline{R}$, with H_0 independent from (ξ,η), so that the maximum points of $\overline{H}$ correspond to the minimum points of $\overline{R}$: from now on, in this proof, we shall work with the averaged perturbing function $\overline{R}$.

$\overline{R}$ is piecewise differentiable in W: the singular sets, composed by points (ξ,η) in each neighborhood of which $\overline{R}$ is not differentiable, correspond to the node crossing lines. If a minimum point is not on a node crossing line, then the averaged perturbing function $\overline{R}$ can be regarded as a Lyapounov function and a neighborhood of the minimum of $\overline{R}$ is a stable region (see [35]).

We shall prove that all the local minima of the function $\overline{R}(\xi,\eta)$ cannot occur neither on the node crossing lines nor on the boundary of W. This proof is divided in two steps

STEP 1: *each local minimum cannot be on a node crossing line.*

We take into account a single ascending node crossing line with some planet with semimajor axis a' (the descending node case is similar): it is defined by the equation $d^+_{nod} = 0$. Let us define the open sets

$$W^+ = int\left\{W\bigcap\{d^+_{nod} > 0\}\right\} \qquad \text{and} \qquad W^- = int\left\{W\bigcap\{d^+_{nod} < 0\}\right\},$$

where $int\{A\}$ means the interior part of a set A.

Given a function $f \in C^\infty(W^+)\bigcap C^\infty(W^-)$ (but not necessarily in $C^\infty(W)$), let

$$\frac{\partial^+}{\partial\xi}f(\xi,\eta) = \frac{\partial}{\partial\xi}f(\xi,\eta)\Big|_{(\xi,\eta)\in W^+}; \qquad \frac{\partial^+}{\partial\eta}f(\xi,\eta) = \frac{\partial}{\partial\eta}f(\xi,\eta)\Big|_{(\xi,\eta)\in W^+};$$

be the partial derivatives of f restricted to W^+, where $d^+_{nod} > 0$, and let

$$\frac{\partial^-}{\partial\xi}f(\xi,\eta)\ ,\ \frac{\partial^-}{\partial\eta}f(\xi,\eta)$$

be the partial derivatives of f restricted to W^-, where $d^+_{nod} < 0$.

If we consider the absolute value of nodal distance $|d^+_{nod}|$ as a function of ξ,η,

$$|d^+_{nod}| = \left|\frac{a(1-\xi^2-\eta^2)}{1+\xi} - a'\right|, \tag{6.1}$$

we obtain

$$\frac{\partial^+}{\partial\xi}|d^+_{nod}| = a\,\frac{\eta^2-\xi^2-2\xi-1}{(1+\xi)^2}, \qquad \frac{\partial^+}{\partial\eta}|d^+_{nod}| = -\frac{2a\eta}{1+\xi}, \qquad (\xi,\eta)\in W^+;$$

and

$$\frac{\partial^-}{\partial\xi}|d^+_{nod}| = -a\,\frac{\eta^2-\xi^2-2\xi-1}{(1+\xi)^2}\ , \qquad \frac{\partial^-}{\partial\eta}|d^+_{nod}| = \frac{2a\eta}{1+\xi}\ , \qquad (\xi,\eta)\in W^-\ .$$

All these functions are uniformly continuous in W^+ and in W^- separately, therefore they can be extended with continuity to the closures $\overline{W^+}$ and $\overline{W^-}$. These extensions generally do not coincide on the node crossing line $d^+_{nod}=0$; indeed they are opposite and in general non zero, in fact we have

$$\begin{cases} \dfrac{\partial^-}{\partial\xi}|d^+_{nod}(\overline{\xi},\overline{\eta})| := \displaystyle\lim_{(\xi,\eta)\to(\overline{\xi},\overline{\eta})}\frac{\partial^-}{\partial\xi}|d^+_{nod}(\xi,\eta)| = -\lim_{(\xi,\eta)\to(\overline{\xi},\overline{\eta})}\frac{\partial^+}{\partial\xi}|d^+_{nod}(\xi,\eta)| := \frac{\partial^+}{\partial\xi}|d^+_{nod}(\overline{\xi},\overline{\eta})|\ ; \\[2ex] \dfrac{\partial^-}{\partial\eta}|d^+_{nod}(\overline{\xi},\overline{\eta})| := \displaystyle\lim_{(\xi,\eta)\to(\overline{\xi},\overline{\eta})}\frac{\partial^-}{\partial\eta}|d^+_{nod}(\xi,\eta)| = -\lim_{(\xi,\eta)\to(\overline{\xi},\overline{\eta})}\frac{\partial^+}{\partial\eta}|d^+_{nod}(\xi,\eta)| := \frac{\partial^+}{\partial\eta}|d^+_{nod}(\overline{\xi},\overline{\eta})|\ ; \end{cases}$$

for each $(\overline{\xi},\overline{\eta})\in\{d^+_{nod}=0\}$.

From Section 4.2 we can deduce that the exterior normal vector N^+ (with unitary length) to the ascending node crossing line is given by

$$N^+ = (N^+_1,N^+_2) = \frac{1}{R}\Big(\xi+\frac{a'}{2a},\eta\Big) = \frac{2a}{(2a-a')}\Big(\xi+\frac{a'}{2a},\eta\Big) \tag{6.2}$$

where R is the radius of the circle representing the ascending node crossing line in the plane (ξ,η).

We notice that if $2a=a'$, the normal vector in (6.2) is not defined: this is a very peculiar case for which the two circles corresponding to the node crossing lines are actually the points $(\xi,\eta)=(-1,0),(1,0)$; it is possible to cross them only if the maximum value allowed for the eccentricity is 1 . When $2a>a'$ there is no crossing between the orbits.

Let us define ∇^+,∇^- as the differential operators $(\partial^+/\partial\xi,\partial^+/\partial\eta)$ and $(\partial^-/\partial\xi,\partial^-/\partial\eta)$ respectively. We take into account the directional derivative along the exterior normal N^+ on the node crossing line: it has an *interior* value, obtained as a limit from values in W^+

$$\frac{\partial^+}{\partial N^+}|d^+_{nod}| = \langle\nabla^+|d^+_{nod}|,N^+\rangle = -\frac{a}{2(1+\xi)^3}\,[1+\xi^2+2\xi-\eta^2]^2-\frac{2a\eta^2}{1+\xi}<0$$

(where $\langle\,,\rangle$ is the Euclidean scalar product), and an *exterior* value exactly opposite

$$\frac{\partial^-}{\partial N^+}|d^+_{nod}| = \langle\nabla^-|d^+_{nod}|,N^+\rangle > 0\ .$$

From the results of Chapter 4 we can compute the averaged perturbing function $\overline{R}$ by

$$\frac{(2\pi)^2}{\mu k^2}\overline{R}(\xi,\eta) = R_1(\xi,\eta) - \frac{2\pi}{\sqrt{det(\mathcal{A})}}d^+_{min}$$

where $R_1(\xi,\eta) \in C^1(\{\xi^2+\eta^2 < 1\})$ and

$$d^+_{min} = |d^+_{nod}| \cdot \left[1 - \frac{a^2a'^2\eta^2}{(1-\xi^2-\eta^2)det(\mathcal{A})}\right]^{\frac{1}{2}} = |d^+_{nod}| \cdot \left[\frac{(1+\xi^2)\sin^2 I}{(1+\xi^2)\sin^2 I+\eta^2}\right]^{\frac{1}{2}} ;$$

as

$$det(\mathcal{A}) = \frac{a^2a'^2[(1+\xi)^2\sin^2 I+\eta^2]}{(1-\xi^2-\eta^2)} .$$

This allows to conclude that the local minima of $\overline{R}(\xi,\eta)$ can not occur on the ascending node crossing line $\{(\xi,\eta) \in W : d^+_{nod} = 0\}$, because the normal derivative of the integral $\overline{R}$ along N^+ is the sum of a smooth function and a discontinuous function positive in W^+ and negative in W^-, so that there is always a side of the node crossing line for which the normal derivative of the restriction to that side of the averaged perturbing function, along N^+ or $-N^+$, is negative.

We can use a similar argument for the descending node crossing lines using the following notation (introduced in Chapter 5):

$$\frac{\partial_1^+}{\partial\xi}f(\xi,\eta)\,; \quad \frac{\partial_1^+}{\partial\eta}f(\xi,\eta)\,; \quad \frac{\partial_1^-}{\partial\xi}f(\xi,\eta)\,; \quad \frac{\partial_1^-}{\partial\eta}f(\xi,\eta)\,;$$

and the differential operators

$$\nabla_1^+ := (\frac{\partial_1^+}{\partial\xi},\frac{\partial_1^+}{\partial\eta})\,; \qquad \nabla_1^- := (\frac{\partial_1^-}{\partial\xi},\frac{\partial_1^-}{\partial\eta})\,.$$

For all the possible choices of the parameters the graphics of $\overline{R}$ in a neighborhood of a node crossing have all a common feature: *they have a crest with the edge towards the higher values* (see Figure 6.4).

As we noticed in section 4.2, a descending node crossing can occur simultaneously with an ascending node crossing, either with the same planet (*double crossing*, see Figure 6.5 $a), c)$) or with another one (*simultaneous crossing*, see Figure 6.5 $b), d)$); In Figure 6.5 $c), d)$ we show cases of *mutually tangent double* and *simultaneous crossings*, that are cases for which the node crossing lines are mutually tangent. These are the only possibilities for multiple crossings in this framework.

When a multiple crossing occurs, we can split the computation of the derivatives of $\overline{R}$, as explained in Chapter 5:

$$\frac{\partial}{\partial e}\int_T \left[\frac{1}{D}\right] d\ell d\ell' = \int_T \frac{\partial}{\partial e}\left[\frac{1}{D} - \frac{1}{d} - \frac{1}{d_1}\right] d\ell d\ell' + \frac{\partial}{\partial e}\int_T \left[\frac{1}{d}\right] d\ell d\ell' + \frac{\partial}{\partial e}\int_T \left[\frac{1}{d_1}\right] d\ell d\ell' ;$$

with this decomposition we have extracted the two singularities appearing due to the ascending and the descending node crossing in the case of double crossing, moving them into the integrals containing only d and d_1; if we have a simultaneous crossing, even if mutually tangent, the two singularities appear in different terms as $1/D$ is a sum of the contributions of each planet separately.

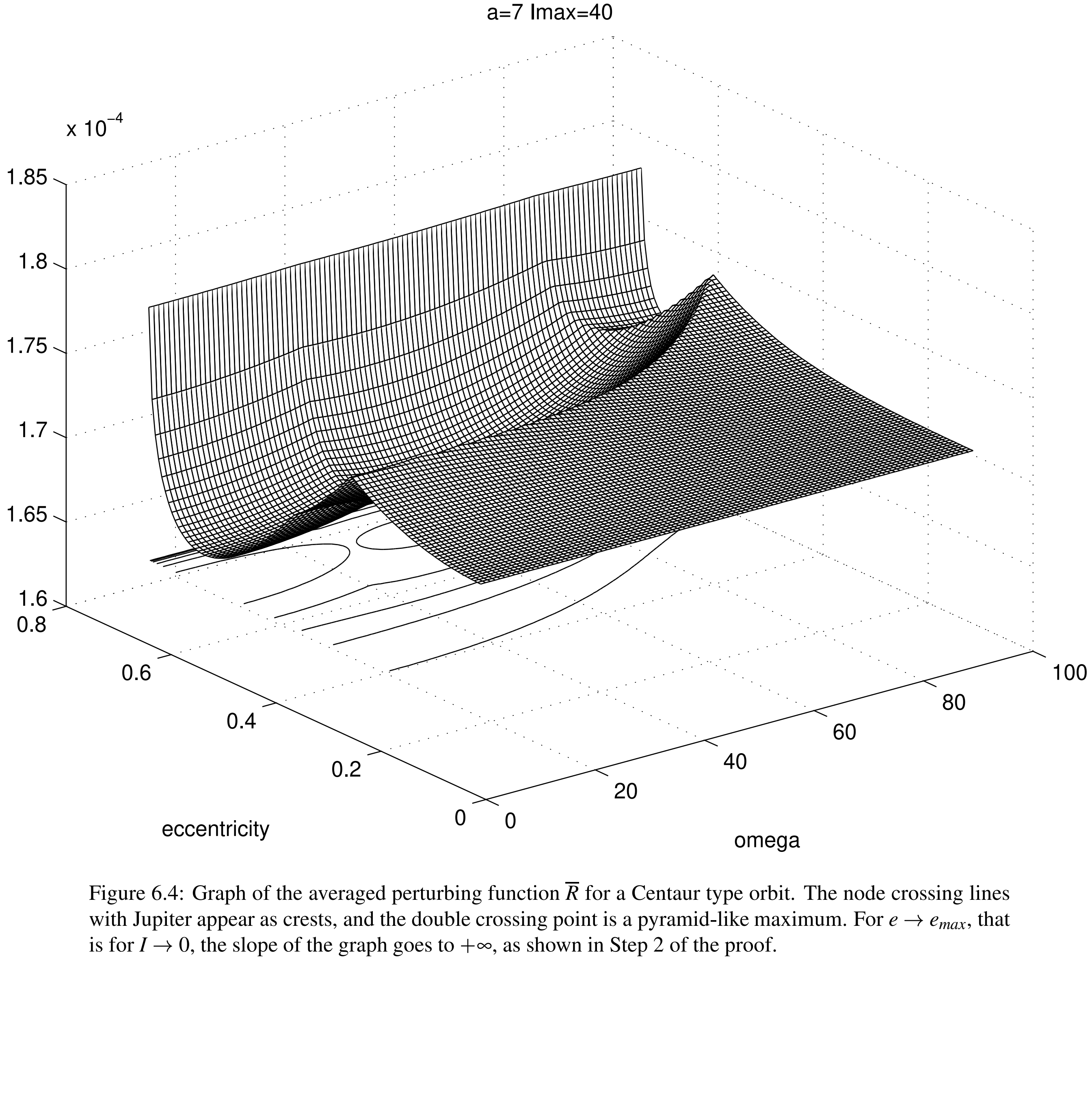

Figure 6.4: Graph of the averaged perturbing function $\overline{R}$ for a Centaur type orbit. The node crossing lines with Jupiter appear as crests, and the double crossing point is a pyramid-like maximum. For $e \to e_{max}$, that is for $I \to 0$, the slope of the graph goes to $+\infty$, as shown in Step 2 of the proof.

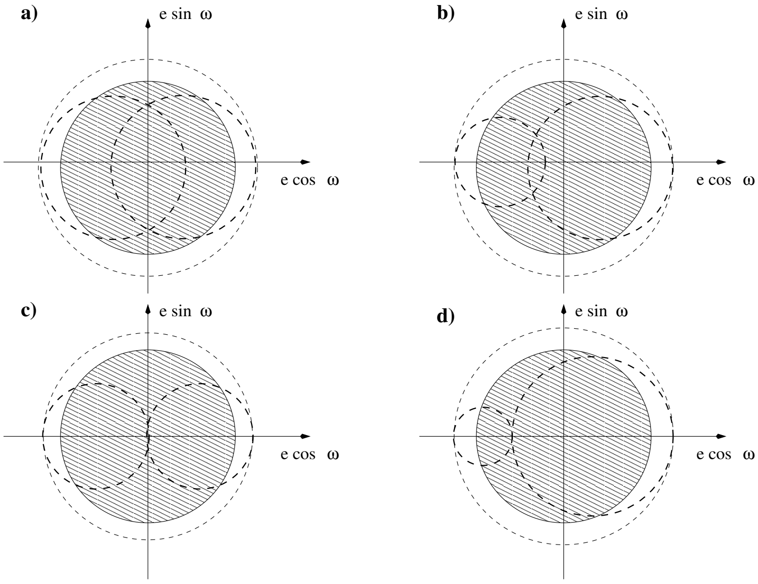

Figure 6.5: Multiple crossings in the framework of the circular coplanar case: *a*) represents a *double crossing*, *b*) is a *simultaneous crossing*, *c*) is a *mutually tangent double crossing* and *d*) is a *mutually tangent simultaneous crossing*. In the last two cases the node crossing lines are tangent to each other.

First we consider the *mutually tangent crossing* case (see *c*) and *d*)). If we compute the normal derivative to the multiple crossing point we can easily deduce that we obtain again a situation like in the single crossing case: there is always a side of a node crossing line for which the limit of one of the two normal derivatives of $\overline{R}$, restricted to that side, to the multiple crossing point, is negative, so that this point cannot be a minimum.

Actually in this case the effect of a multiple crossing is to increase the discontinuity of the derivative caused by each node crossing separately.

We can use the results on the normal derivatives also when dealing with a double crossing that is *not* mutually tangent. Let us consider one of the two intersection points, say P: we draw in Figure 6.6 the exterior normal unit vector N_P^+ (with unitary length) and the tangent unit vector T_P^+ to the ascending node crossing (counter-clockwise) circle $\{d_{nod}^+ = 0\}$; correspondingly we also draw the normal and tangent vectors N_P^-, T_P^- to the descending node crossing circle $\{d_{nod}^- = 0\}$.

The vectors T_P^+ and T_P^- are linearly independent and T_P^+ can be brought onto T_P^- by a counter clockwise rotation of an angle $\theta \in]0,\pi[$ around P, so that we can find $\alpha, \beta \in \mathbb{R}$ with $\beta < 0$ such that

$$T_P^- = \alpha T_P^+ + \beta N_P^+.$$

We compute now the limits of derivatives of $\overline{R}$ along the vector T_P^- at the point P, coming from internal and external values to the curve $\{d_{nod}^+ = 0\}$: We have

$$\frac{(2\pi)^2}{\mu k^2}\overline{R}(\xi,\eta) = R_1(\xi,\eta) - \frac{2\pi}{\sqrt{det(\mathcal{A})}}\,d_{min}^+ - \frac{2\pi}{\sqrt{det(\mathcal{A}_d)}}\,d_{min}^-$$

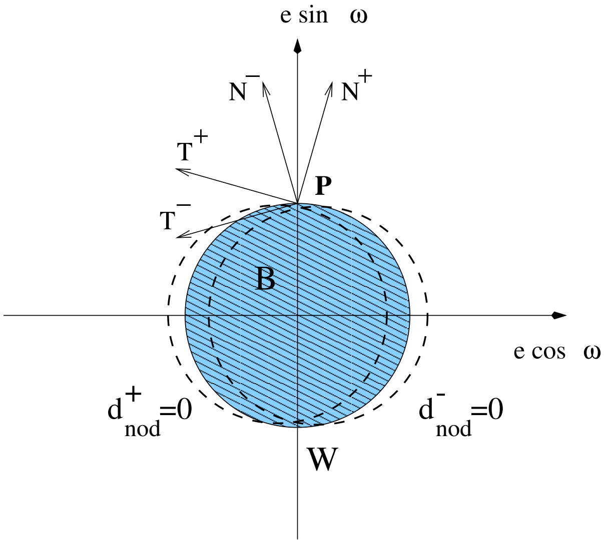

Figure 6.6: Node crossing lines for a double crossing that is *not* mutually tangent. The normal and tangent vectors to both curves are also drawn.

where $R_1(\xi,\eta) \in C^1(\{\xi^2+\eta^2<1\})$ and d^+_{min}, d^-_{min} are the minimal distances between the straight lines that approximate the conics at the ascending and descending nodes.

As $\overline{R}_1(\xi,\eta)$ is a regular function, we have

$$\frac{\partial^+}{\partial T_P^-}R_1(P) = \frac{\partial^-}{\partial T_P^-}R_1(P) = \frac{\partial}{\partial T_P^-}R_1(P) \; .$$

It follows that

$$\begin{aligned}
\frac{(2\pi)^2}{\mu k^2}\frac{\partial^+}{\partial T_P^-}\overline{R}(P) &= \frac{\partial^+}{\partial T_P^-}R_1(P) - \frac{2\pi}{\sqrt{(det(\mathcal{A})}}\frac{\partial^+}{\partial T_P^-}d^+_{min}(P) - \frac{2\pi}{\sqrt{(det(\mathcal{A}_d)}}\frac{\partial^+}{\partial T_P^-}d^-_{min}(P) = \\
&= \frac{\partial}{\partial T_P^-}R_1(P) - \frac{2\pi}{\sqrt{det(\mathcal{A})}}\lim_{(\xi,\eta)\to P}\langle\nabla^+ d^+_{min}(\xi,\eta)\,,\,T_P^-\rangle - \frac{2\pi}{\sqrt{det(\mathcal{A}_d)}}\frac{\partial^+}{\partial T_P^-}d^-_{min}(P) = \\
&= \frac{\partial}{\partial T_P^-}R_1(P) - \frac{2\pi}{\sqrt{det(\mathcal{A})}}\lim_{(\xi,\eta)\to P}\langle\nabla^+ d^+_{min}(\xi,\eta)\,,\,\alpha T_P^+ + \beta N_P^+\rangle - \frac{2\pi}{\sqrt{det(\mathcal{A}_d)}}\frac{\partial^+}{\partial T_P^-}d^-_{min}(P)
\end{aligned}$$

where

$$\lim_{(\xi,\eta)\to P}\langle\nabla^+ d^+_{min}(\xi,\eta)\,,\,\alpha T_P^+ + \beta N_P^+\rangle = \alpha\frac{\partial^+}{\partial T_P^+}d^+_{min}(P) + \beta\frac{\partial^+}{\partial N_P^+}d^+_{min}(P) \; .$$

We notice that

$$\begin{aligned}
\frac{\partial^+}{\partial T_P^-}d^-_{min}(P) &:= \lim_{(\xi,\eta)\to P}\frac{\partial^+}{\partial T_P^-}d^-_{min}(\xi,\eta)) = \lim_{(\xi,\eta)\to P}\frac{\partial}{\partial T_P^-}d^-_{min}(\xi,\eta)\Big|_{(\xi,\eta)\in B} = \\
&= \lim_{(\xi,\eta)\to P}\frac{\partial_1^+}{\partial T_P^-}d^-_{min}(\xi,\eta) := \frac{\partial_1^+}{\partial T_P^-}d^-_{min}(P)
\end{aligned} \tag{6.3}$$

where $B = \{d^+_{nod} > 0\} \cap \{d^-_{nod} > 0\} \cap W$.

We can also prove that

$$(i) \quad \frac{\partial^+}{\partial T^+_P} d^+_{min}(P) = 0 ; \qquad\qquad (ii) \quad \frac{\partial^+_1}{\partial T^-_P} d^-_{min}(P) = 0 . \tag{6.4}$$

Property (i) in equations (6.4) holds because

$$d^+_{min} = |d^+_{nod}| \left[\frac{(1+\xi)^2 \sin^2 I}{(1+\xi)^2 \sin^2 I + \eta^2} \right]^{\frac{1}{2}} ;$$

and

$$\frac{\partial^+}{\partial T^+_P} |d^+_{nod}(P)| = \lim_{(\xi,\eta)\to P} \frac{\partial^+}{\partial T^+_P} |d^+_{nod}(\xi,\eta)| = 0$$

as we can easily verify using the relations

$$T^+_P = \frac{2a}{2a-a'} \left(\eta, -\left(\xi + \frac{a'}{2a} \right) \right) ; \qquad \nabla^+ |d^+_{nod}| = \left(\frac{a(\eta^2 - \xi^2 - 2\xi - 1)}{(1+\xi)^2} \; , \; -\frac{2a\eta}{1+\xi} \right) ;$$

and

$$1 + \xi = \frac{a}{a'} (1 - \xi^2 - \eta^2) .$$

Property (ii) in equations (6.4) can be shown in a similar way. Then we have

$$\frac{(2\pi)^2}{\mu k^2} \frac{\partial^+}{\partial T^-_P} \overline{R}(P) = \frac{\partial}{\partial T^-_P} R_1(P) - \frac{2\pi}{\sqrt{det(\mathcal{A})}} \beta \frac{\partial^+}{\partial N^+_P} d^+_{min}(P) .$$

By a similar argument we obtain

$$\frac{(2\pi)^2}{\mu k^2} \frac{\partial^-}{\partial T^-_P} \overline{R}(P) = \frac{\partial}{\partial T^-_P} R_1(P) - \frac{2\pi}{\sqrt{det(\mathcal{A})}} \beta \frac{\partial^-}{\partial N^+_P} d^+_{min}(P) .$$

As the normal derivatives of $|d^+_{nod}|$ are opposite, if computed from both sides of the ascending node crossing line, we can use the same argument of the single crossing to conclude.

The case of the *not mutually tangent simultaneous crossing* is similar. We conclude that a minimum of $\overline{R}$ can not occur on any point belonging to one or more crossing lines.

STEP 2: *the minimum cannot be on the boundary of W .*

To exclude the possibility of a minimum along the boundary $e = e_{max}$ we are going to prove that the directional derivative along the exterior normal to the boundary, on a surface $Z = const$, is positively divergent for $e \to e_{max}$:

$$\lim_{e\to e_{max}} \frac{\partial}{\partial e} \left[\overline{R}(e,\omega) \Big|_{Z=const} \right] = +\infty$$

for almost all ω. For $e = e_{max}$, that is for $I = 0$, the averaged perturbing function does not depend on ω for symmetry reasons, and its value is then a local maximum, thus it can not be a minimum.

The computation of the normal derivative to the boundary corresponds to the total derivatives with respect to the eccentricity e taking into account the relation between the inclination I and e given by the constant value of the integral Z, that is the total derivative of the restriction to the manifold $Z = const$:

$$\frac{\partial}{\partial e}\left[\overline{R}(e,\omega)\Big|_{Z=const}\right] = \frac{\partial}{\partial e}\overline{R} - \frac{\cos I}{\sin I}\,\frac{e}{(1-e^2)}\,\frac{\partial}{\partial I}\overline{R}\ . \tag{6.5}$$

The derivative with respect to the inclination of the inverse distance $1/D$ can be computed by

$$\frac{\partial}{\partial I}\left[\frac{1}{D}\right] = -\frac{1}{2D^3}\frac{\partial}{\partial I}D^2\ ; \tag{6.6}$$

for $I = 0$, that is $e = e_{max}$, this expression has a polar singularity of order ≤ 3; we will show that it is indeed of order 3. This occurs whenever there is orbit crossing.

If we consider $I \neq 0$, we can write

$$\frac{1}{\sin I}\,\frac{\partial}{\partial I}D^2 = 2aa'\sin u'[(\cos u - e)\sin\omega + \sqrt{1-e^2}\,\sin u\cos\omega]\ .$$

We derive now the relations between the anomalies $\overline{u},\overline{u}'$ at the crossing points for $I = 0$. Let ξ,η be the first two variables of the reference used in Chapter 4: if we consider a reference frame ξ',η' with the ξ' axis along the pericenter of the orbit of the asteroid, as in Figure 6.7, and we set the coordinates corresponding to the circle and to the ellipse to be equal in this reference frame, we get

$$a(\cos\overline{u} - e) = a'\cos(\overline{u}' - \omega)\ ; \qquad a\sqrt{1-e^2}\,\sin\overline{u} = a'\sin(\overline{u}' - \omega)\ . \tag{6.7}$$

If the plane crossing condition

$$a(1-e) < a' < a(1+e)$$

holds, then there are two pairs of values of the anomalies that satisfy equations (6.7). If $a(1-e) = a'$ or $a(1+e) = a'$, then there is a *tangent crossing* and there is only one pair of values of the anomalies that corresponds to the crossing condition.

The angles measured from the ξ'-axis and the ones from the ξ-axis differ by the argument of perihelion ω, that is measured from the same origin as the anomaly u'.

We notice that if we arrive at $e = e_{max}$ passing through node crossing values (see Figure 6.7) then we obtain a polar singularity of order one, as we have already observed in Chapter 5.

At the crossing points $\overline{u},\overline{u}'$, defined by equations (6.7), we obtain that

$$\begin{aligned}\frac{1}{\sin I}\,\frac{\partial}{\partial I}D^2(\overline{u},\overline{u}') &= 2aa'\sin\overline{u}'\left[\frac{a'}{a}\cos(\overline{u}'-\omega)\sin\omega + \frac{a'}{a}\sin(\overline{u}'-\omega)\cos\omega\right] = \\ &= 2(a')^2\sin^2\overline{u}'\end{aligned}$$

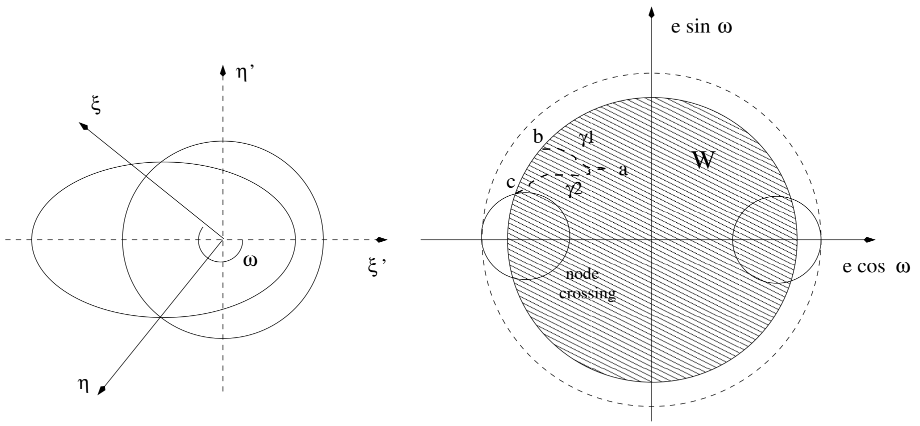

Figure 6.7: On the left we draw the picture of the (ξ',η') reference frame, useful to write the relations between $\overline{u},\overline{u}'$ at crossing. On the right we can see two different paths to the value $e = e_{max}$.

is always positive, with the only exception of the case $\overline{u}' = 0,\pi$, corresponding to the intersection of the node crossing lines with the boundary $e = e_{max}$. Thus the right hand side of equation (6.6) has a polar singularity of order 3 occurring for $I = 0$ and for the values of $\overline{u},\overline{u}'$ satisfying (6.7); moreover, it is positive in a neighborhood of the singularity.

The partial derivative with respect to e of $1/D$, appearing in equation (6.5), has a polar singularity of order two at most, in fact, setting $I = 0$ in the expression of $\frac{\partial}{\partial e}\frac{1}{D}$ (see Appendix B) , we obtain that

$$\frac{\partial}{\partial e}\left[\frac{1}{D}\right] = -\frac{1}{2D^3}\frac{\partial}{\partial e}D^2;$$

$$\frac{\partial}{\partial e}D^2 = -2a^2(\cos u - e) - 2ea^2\sin^2 u + 2aa'\Big\{\cos u'\cos\omega -$$

$$-\frac{e\sin\omega}{\beta}\sin u\cos u' + \sin\omega\sin u' + \frac{e\cos\omega}{\beta}\sin u\sin u'\Big\} =$$

$$= -2a^2(\cos u - e) - 2ea^2\sin^2 u + 2aa'\Big\{\cos(u'-\omega) - \frac{e}{\beta}\sin u\sin(u'-\omega)\Big\}\,.$$

If we evaluate this quantity at $\overline{u},\overline{u}'$, using relationships (6.7) we obtain

$$\frac{\partial}{\partial e}D^2(\overline{u},\overline{u}') = 0;$$

so, if we develop in Taylor's series the e-derivative of D^2 in a neighborhood of $\overline{u},\overline{u}'$ we have

$$\left|\frac{\partial}{\partial e}D^2\right| \le O\left(\sqrt{|u-\overline{u}|^2 + |u'-\overline{u}'|^2}\right)$$

while $1/D^3$ has a third order polar singularity, so $\partial R/\partial e$ has a second order polar singularity at most.

Thus the term coming from the I derivative of $1/D$, that gives the pole of order 3, is the principal part of the right hand side of equations (6.5) for $I = 0$, and the following integral is positively divergent:

$$\int\int_T \frac{\partial}{\partial e}\left[R(e_{max},\omega;\ell,\ell')\Big|_{Z=const}\right] d\ell\, d\ell' = +\infty \,.$$

We can conclude using Fatou's lemma (see [8]): let $P_{e,\omega}(\ell,\ell')$ be the principal part of the function

$$\frac{\partial}{\partial e}\left[R(e,\omega;\ell,\ell')\Big|_{Z=const}\right]$$

for $e \to e_{max}$; then there is a neighborhood U of the singularity where the functions P_e are positive, and where

$$\frac{\partial}{\partial e}\left[R(e,\omega;\ell,\ell')\Big|_{Z=const}\right] \geq \frac{1}{2}\, P_{e,\omega}(\ell,\ell') \,.$$

Applying Fatou's lemma we have

$$\liminf_{e\to e_{max}} \int\int_U \frac{1}{2} P_{e,\omega}(\ell,\ell')\, d\ell\, d\ell' \geq \int\int_U \liminf_{e\to e_{max}} \frac{1}{2} P_{e,\omega}(\ell,\ell')\, d\ell\, d\ell' \,,$$

with the latter integral positively divergent. We conclude that

$$\liminf_{e\to e_{max}} \int\int_U \frac{\partial}{\partial e}\left[R(e,\omega;\ell,\ell')\Big|_{Z=const}\right] d\ell\, d\ell' \geq \liminf_{e\to e_{max}} \int\int_U \frac{1}{2} P_{e,\omega}(\ell,\ell')\, d\ell\, d\ell' = +\infty \,.$$

This argument can be repeated for the other intersection point between the two conics occurring for $I = 0$ if there is no tangent crossing, and for its neighborhood U_1.

The integrals I_e on the remaining part of the torus, $T - U - U_1$,

$$I_e = \int\int_{T-U-U_1} \frac{\partial}{\partial e}\left[R(e,\omega;\ell,\ell')\Big|_{Z=const}\right] d\ell\, d\ell'$$

are uniformly limited with respect to e, at least in a (left) neighborhood of e_{max}, thus they do not change the infinite value of the limit. This concludes the proof of the theorem.

□

Chapter 7

Secular evolution theory and proper elements for NEAs

In this chapter we describe a method to compute the secular evolution of Near Earth Asteroids in the framework of a Solar System with the planets on circular coplanar orbits. This method is based on the generalized averaging principle described in Chapters 4 and 5 and it has been introduced in [54], [30].

Later we shall give the definition of proper elements for NEAs, that are quantities approximatively constant during the motion, and we shall explain how to compute them using the secular evolution theory.

7.1 The secular evolution algorithm

Recall that the averaged Hamiltonian $\overline{H}$ *is invariant under the symmetries*

$$\omega \longrightarrow -\omega; \qquad \omega \longrightarrow \pi - \omega; \qquad \omega \longrightarrow \pi + \omega;$$

this allows to draw the level lines of the averaged Hamiltonian, on which the solution curves are confined, simply knowing their shape in a subset of the reduced phase space (ω, e) of the form

$$\{(\omega, e) : k\pi/2 \leq \omega \leq (k+1)\,\pi/2\ ,\ 0 \leq e \leq e_{max}\}$$

with $k \in \mathbb{Z}$.

The numerical method that we use to solve the averaged Hamilton's equations (4.3) is an implicit symplectic Runge-Kutta-Gauss algorithm of order 6 (see [65]).

Kantorovich's method has a key role in writing the secular evolution algorithm: it is used for the following purposes:

1. to compute the analytic expression of the terms of the derivatives of $\overline{H}$ that are discontinuous on the node crossings lines;

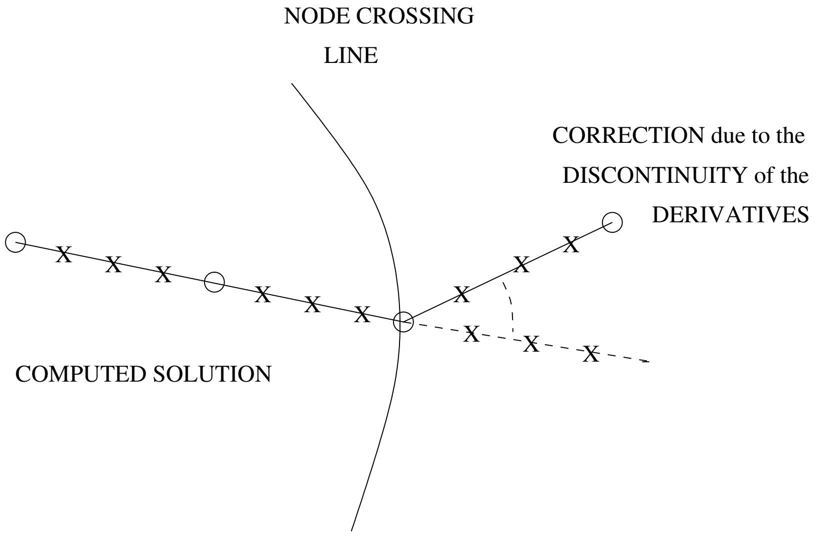

Figure 7.1: Graphical description of the secular evolution algorithm when a crossing occurs: we use an implicit symplectic Runge-Kutta-Gauss method of order 6. In an integration step, delimited by two consecutive circles, the only points in which the derivatives of $\overline{R}$ are computed are the 3 intermediate points marked with crosses. When the node crossing line is reached, at the end of one integration step, a correction taking into account the explicitly computed discontinuity of the derivatives is added to the right hand sides of the equations.

2. to stabilize the computations when an orbit is very close to a node crossing, the nodal distance is small and the integrand functions have very sharp extrema.

The right hand side of equations (4.3) are never computed at the node crossing points, where they are defined in a twofold way. This is possible because the Runge-Kutta-Gauss methods use sub-steps which include neither the starting point nor the final point of the step being computed. We resort to the following procedure (which had already been used in [54]): every time the asteroid orbit is close enough to a node crossing line, the standard iteration scheme known as *regula falsi* is used to set the second extreme point of the step exactly on that line; as the computation of the right hand sides is performed only in the intermediate points, the computation at node crossings is avoided.

When the node crossing point is reached within the required precision, the integration needs to restart from there for the next step. In the implicit Runge-Kutta schemes the following step restarts the iterations (to solve the implicit equations) from a first guess, obtained by polynomial extrapolation, of the values used in the previous step. This would fail in this case, because of the discontinuity of the right hand side of the averaged equations (see Figure 7.1); we apply to the first guess, extrapolated from values computed before the node crossing, a correction computed by means of the explicit analytical formulas for the discontinuity of the derivatives of $\overline{R}$ (see Chapter 5).

There are several technicalities involved in the regula falsi method to arrive at the node crossing line exactly at the end of one integration step. This is necessary to ensure that the computational procedure is sufficiently stable.

The main problem is that, the right hand side being discontinuous on the node crossing line, the integration steps need to be controlled to be rather shorter than longer than the exact time span to node crossing, to avoid running into the discontinuous behavior.

The regula falsi iterations can start only when a change in the sign of the nodal distance takes place between the last intermediate point of the step and the beginning of the next step (with the symbols used in Figure 7.1, the node crossing line must be between the last cross and the next circle). If a change of sign already occurs in the intermediate points, the step-size is halved. Even when no change of sign occurs in the current step, an extrapolation algorithm is used to predict whether a node crossing could take place in the next step. For this purpose, the step-size to crossing is computed in two ways, by regula falsi, that is by a linear extrapolation, and by a polynomial extrapolation; the minimum of the two is taken as a prudent estimate of the step-size to crossing. If the step-size to crossing is less than the step-size used when there are no crossings, the step-size of the integration is modified using this prediction, in order to avoid *overshooting* of the node crossing line.

A separate regula falsi algorithm is used to compute the value of the orbital elements solution of the averaged equations exactly at the symmetry lines (which corresponds to integer multiples of $\omega = \pi/2$); this is used for the computation of the proper elements, as will be explained in Section 7.3.

We have already noticed in Chapter 4 that the periodicity of the function $D(u,u')$, representing the distance between two points on the two osculating ellipses of a planet and of the asteroid is used in order to keep the singular points inside the integration domain. The translated domains used in the FORTRAN program of the secular evolution algorithm are represented in Table 7.1

NO CROSSING	$\{(\ell,\ell') : 0 \leq \ell < 2\pi; -\pi \leq \ell' < \pi]\}$
ASCENDING NODE CROSSING	$\{(\ell,\ell') : 0 \leq \ell < 2\pi; -\pi \leq \ell' < \pi]\}$
DESCENDING NODE CROSSING	$\{(\ell,\ell') : 0 \leq \ell < 2\pi; 0 \leq \ell' < 2\pi]\}$
DOUBLE CROSSING	$\{(\ell,\ell') : -\pi \leq \ell < \pi; -\pi/2 \leq \ell' < 3/2\pi]\}$

Table 7.1: Shifts of the integration domain $\mathbb{T}^2$ according to the type of node crossing occurring during the integration.

7.2 Different dynamical behavior of NEAs

We have used the secular evolution algorithm, introduced in Section 7.1, to compute the averaged evolution of all the NEAs known so far. Here we describe the main features of the secular evolution of these asteroids.

7.2.1 Classification of NEAs

The NEAs are traditionally divided into three families:

1. *Athens* (after asteroid (2062) Athen)

$$a < 1.0 \text{ AU}; \qquad Q = a(1+e) > 0.983 \text{ AU};$$

2. *Apollos* (after asteroid (1862) Apollo)

$$a > 1.0 \text{ AU}; \qquad q = a(1-e) < 1.017 \text{ AU};$$

3. *Amors* (after asteroid (1221) Amor)

$$a > 1.0 \text{ AU}; \qquad q = a(1-e) = 1.017 \text{ to } 1.3 \text{ AU} .$$

The classification in Apollo, Amor and Athen is generally not stable for time spans of $100, 1\,000$ years or more [50] and we shall see in Section 7.3.3 that the proper elements can be used to find the NEAs for which this classification is not valid for such times.

7.2.2 Qualitative behavior of the orbits

In our integration we have detected two of the main qualitative behaviors in the (ω, e) described in Chapter 6: *direct circulation* and *counter-clockwise symmetric libration* of the perihelion argument. No *asymmetric libration* (that is a libration whose center is not an integer multiple of $\omega = \pi/2$) has been found with this set of orbital elements, although we know it could occur e.g. for comet/Centaur type orbits as we have seen in Figure 6.3.

In Figures 7.2, 7.3 there is a sample of the secular evolution of the NEAs in the (ω, e) plane; the level lines of the averaged perturbing function $\overline{R}$ are plotted and the time evolution of the objects is enhanced by crosses along the curve representing the constant value of $\overline{R}$.

In Figure 7.2 there are four examples of circulation of ω. The first object, 1991 TT is an Apollo asteroid that is preserved from Earth crossings by an $e-\omega$ coupling mechanism; the second asteroid, 1991 BB, is an ordinary quadruple crosser (it crosses the Earth four times for period of revolution of ω) while the third, 1994 JX, is an Amor asteroid that becomes crosser during its evolution (it is even an octuple crosser). The last case, 1998 FG_{12}, is peculiar because it crosses the Earth orbit in a tangent way, remaining in a neighborhood of the crossing curve for a very long time.

In Figure 7.3 there are four examples of librating objects. The first is an Athen asteroid preserved from crossing the Earth orbit by a symmetric libration around $\omega = 2\pi$; on the other hand the libration in the second example, (2102) Tantalus, forces this asteroid to cross the orbits of Venus and the Earth, both four times for each period of the libration argument.

The third example is 1999 AN_{10}, whose dynamics has been extensively studied because of its very small distance from the orbit of the Earth at both nodes: when fewer observations were available, it was not possible to exclude an impact with our planet in the next century. As Figure 7.3 shows, the secular evolution is now very close to a double crossing at both the ascending and descending node with the Earth; the period of libration

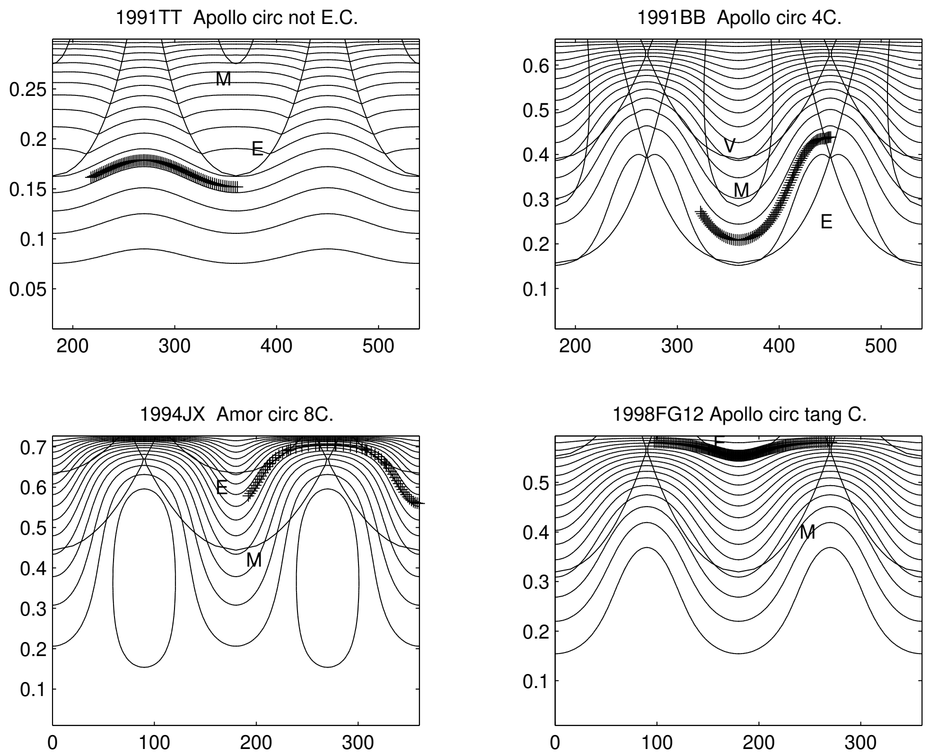

Figure 7.2: Time evolution of NEAs whose perihelion argument is circulating. The node crossing lines with the Earth (E), Mars (M) and in some cases with Venus (V) are also shown. The averaged evolution has been followed for a time span depending on the precession rate; the shortest time span is about 3500 *yrs* for 1994*JX* and the longest is about 45000 *yrs* for 1991*BB*.

of ω is very long, and this implies that the dangerous closeness of this orbit to the orbit of the Earth will be maintained for centuries.

The last asteroid of Figure 7.3, 1997 NC_1, has a very strange averaged dynamics: the level curve of the averaged Hamiltonian, to which its motion is bounded, is very close to the separatrix between libration and circulation, so that, while its perihelion argument in the long run circulates, its orbit draws an arc of libration near the integer multiples of π.

7.3 Proper elements

We give the definition of proper elements for Near Earth Asteroids, and we explain how to compute them using the generalized averaging principle presented in Chapters 4, 5.

We have computed the proper elements for all the known Near Earth Asteroids: they are continuously updated and are available in a catalog at the web address

`http://newton.dm.unipi.it/neodys/` .

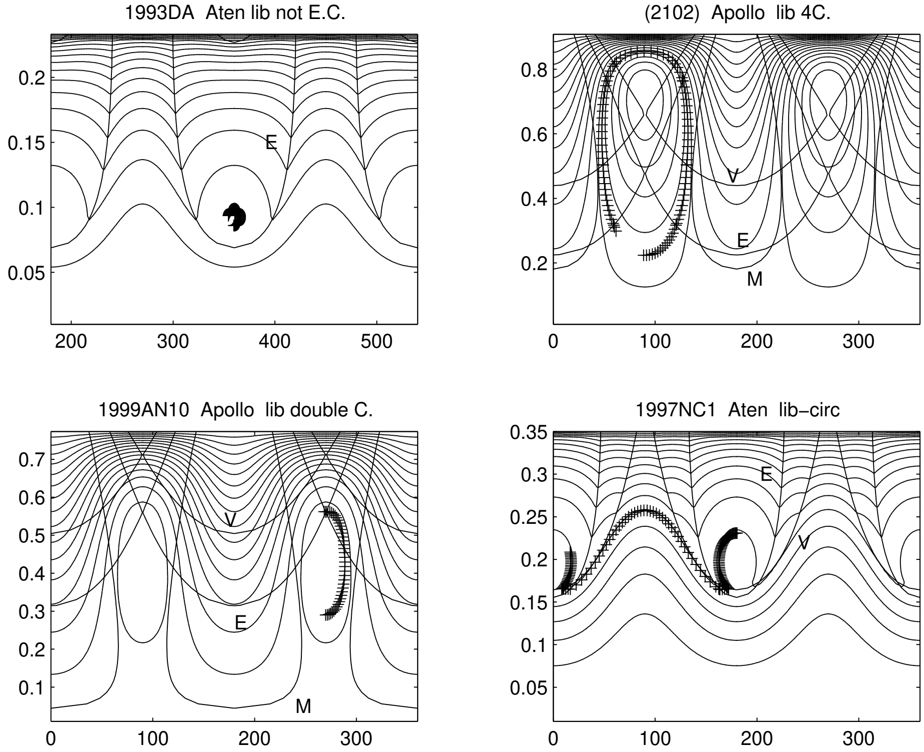

Figure 7.3: Time evolution of Near Earth Asteroids whose perihelion argument is librating. In this case the time spans is between about 24000 *yrs* for $1999AN_{10}$ and about 60500 *yrs* for $1997NC_1$.

7.3.1 History of proper elements

In the study of Dynamics a very important role is played by the integrals of the motion, that is by quantities that are constant during the time evolution of a dynamical system. When the dynamics is non integrable, as it is the case for the N-body problem ($N \geq 3$), it is also useful to compute quantities that are approximatively constant during the motion. We give the following

DEFINITION 19. *Given a dynamical system, we call* proper elements *each first integral of a system that approximates the real motion of the original problem.*

With this definition the proper elements are quasi–integrals of the motion, that is quantities that change very slowly with time and can be considered approximatively constant over time spans not too long.

The first to employ the concept of *proper elements* was Hirayama [32], [33], [34]: he used a linear theory to identify asteroid families in the main belt. The identification of families together with the understanding of the dynamical structure of the asteroid belt (for instance of the relevance of secular resonances) are two important reasons for the computation of proper elements for Main Belt Asteroids (MBAs).

There are presently three different methods to compute these quantities:

1. an analytical theory by Milani and Knežević [55], [56], [57]: it is based on series expansion in eccentricity and inclination, particularly suitable for low inclination ($< 17°$) and low eccentricity orbits. The proper elements computed in [56], [57] have been proven to be stable over time scales of the order of 10^7 years.

2. a semianalytic theory by Lemaître and Morbidelli [41]: it is more appropriate for the orbits with either large eccentricities or large inclinations; it is based on the classical averaging method in its revisited version [31]. A similar method had already been used by Williams to obtain a set of proper elements that led to the understanding of the secular resonances resulting from two secular frequencies being equal [78], [79], [80].

3. a synthetic theory by Knežević and Milani [38]: it is based on the computation of the asteroid orbits by pure numerical integrations; the short periodic perturbations are then removed by a filtering process performed during the integration. This recent theory allows to obtain a high accuracy of the elements; on the other hand it requires CPU times longer than the two preceding methods.

Almost all the NEAs are planet crossing, and the singularities coming from the possibility of collision results in the strong divergence of the series used in the analytical theory and in the divergence of the integrals of the classical averaging principle used in the semianalytic theory. Furthermore the strong chaoticity of crossing orbits and the very short integration steps to be chosen in this case make unappliable the synthetic theory on large scale: in fact we have to use several values of the initial phase on the orbit for each NEA and in this case it would require very long computational times.

On the other hand the reasons to compute proper elements are very different in the case of NEAs and the required stability times are much shorter: for most applications, time spans between 10 000 and 100 000 years are long enough. Over longer time spans the dynamics is dominated by large changes in orbital elements, including the semimajor axis, resulting from close approaches with the planets and from the effects of secular resonances.

The reasons to compute proper elements for NEAs are mainly the following:

1. to detect the possibility and compute the probability of collision of Earth-crossing objects with the Earth over intermediate time scales (thousands to tens of thousands years);

2. to identify the objects whose long term orbital evolution is controlled by one or more of the main secular resonances and to draw a map of the dynamical structure of the Earth crossing region in the phase space;

3. to identify meteor streams (sets of very small objects that can be observed only when they are crossing the orbit of the Earth) and to give a criterion to be used in the identification of their parent bodies, for example using suitable variables as in [74].

Even if in most cases an averaged integration requires a CPU time shorter than the one required to integrate the full equations, both these computations can be done in a short time with modern computers. On the other hand, when a NEA is close to a node crossing, then the predictability horizon of its orbit is at most of the order of a few tens of years; hence if we wish to compute, by full numerical integrations, proper elements that have to be reliable over longer a time-span, we have to perform the computation in a statistical sense, that is we have to use a large sample of initial conditions (sampling the mean anomaly over an entire period, as it is the most uncertain element) and we should do a large number of numerical integrations for each asteroid. This procedure would require CPU times too long even with the recent computers; furthermore we have to take into account that the number of the known NEAs increases more and more rapidly, thanks to observational efforts and technology improvements.

A valid method allowing to compute in short time proper elements for NEAs that are reliable over time-spans between 1000 and 100 000 years can thus provide a good tool for their study.

7.3.2 Definition of proper elements and proper frequencies

In this Section we shall define the proper elements for the small celestial bodies moving in a planetary system with the planets on circular coplanar orbits and we shall describe the algorithm to compute them.

Given the osculating orbit of a NEA, represented by its Keplerian elements $(a, e, I, \omega, \Omega)$ at a given time t, the following quantities are constant during the averaged motion in the framework of the circular coplanar case (see Chapters 4, 5):

$$a, e_{min}, e_{max}, I_{min}, I_{max} \ ;$$

they are respectively the semimajor axis and the minimal and maximal value of the averaged eccentricity and the averaged inclination.

We consider as set of proper elements either

$$\{a, e_{min}, I_{max}\} \qquad \text{or} \qquad \{a, e_{max}, I_{min}\} \ .$$

Remark 17. If we consider an ω-librating orbit we can also define

$$\omega_{min}, \omega_{max},$$

that are the minimal and maximal value of the averaged perihelion argument. These quantities are also constant during the averaged motion and can give additional informations to understand the dynamics of the objects that we are studying.

We can also use a set of proper elements in which the extreme values of the eccentricity and inclination are substituted by the secular frequencies of the longitude of perihelion g and the longitude of the node s in case of ω-circulation. If ω is librating we can use the libration frequency l_f in place of g.

The computation of these proper frequencies requires some additional comments:

Proposition 8. *If* ω *is* circulating, *then let* t_0 *and* t_1 *be the times of passage at* 0 *and* $\pi/2$ *and let* $\Omega_{t_0}, \Omega_{t_1}$ *be the corresponding values of the longitude of the node. We have the following formulas to compute the secular frequencies of the argument of perihelion* $g-s$ *and of the longitude of the node* s*:*

$$g-s = \frac{2\pi}{4(t_1-t_0)}; \qquad s = \frac{\Omega_{t_1}-\Omega_{t_0}}{t_1-t_0} \ .$$

If ω *is* symmetrically librating, *then let* τ_0 *and* τ_1 *be two consecutive times of passage at the same integer multiple of* $\pi/2$ *and let* $\Omega_{\tau_0}, \Omega_{\tau_1}$ *be the corresponding values of the longitude of the node. We can compute the frequency of the longitude of the node* s *and the libration frequency* l_f *by the following formulas:*

$$s = \frac{\Omega_{\tau_1}-\Omega_{\tau_0}}{\tau_1-\tau_0}; \qquad l_f = \pm\frac{2\pi}{2(\tau_1-\tau_0)};$$

where the sign has to be chosen negative for clockwise libration.

Proof. The formula for $g-s$ is an immediate consequence of the fact that the period of circulation of ω must be four times the time interval required by an increase by $\pi/2$ of ω.

The proper frequency s has to be computed taking into account that the node has a secular precession, but also long periodic oscillations controlled by the argument 2ω.

The averaged Hamiltonian does not contain Ω because of the invariance with respect to rotation around the z axis; according to D'Alembert's rules (see [72]) it contains only the cosine of 2ω, thus the perturbative equation of motion for Ω is

$$\frac{d\Omega}{dt} = -\frac{\partial\overline{R}}{\partial Z}(2\omega) \ .$$

If ω is circulating it is possible to change (at least locally) variable and write the equation

$$\frac{d\Omega}{d\omega} = \frac{\partial\overline{R}/\partial Z}{\partial\overline{R}/\partial G} = F(\cos(2\omega))$$

where F is an even function of $\cos(2\omega)$. The solution for Ω as a function of ω is:

$$\Omega(\omega) = \alpha\,\omega + f(\sin(2\omega))$$

where $f(\sin(2\omega))$ is zero for $\omega = 0, \pi/2$, as f is a continuous odd function of $\sin(2\omega)$; thus

$$\Omega_{t_1} - \Omega_{t_0} = \alpha\,\pi/2$$

identifies the secular part of the evolution of Ω, whose frequency can be computed by

$$s = \frac{\Omega_{t_1}-\Omega_{t_0}}{t_1-t_0} \ .$$

The symmetric libration cases are different because ω returns to the same multiple of $\pi/2$ after a time interval $\tau_1-\tau_0$, while in the same time span Ω has changed by $\Omega_{\tau_1}-$

Ω_{τ_0}. Then $g-s$ is not a proper frequency (its secular part is by definition zero), and by arguments that are similar to the ones above, the libration frequency l_f and the secular frequency of the node s can be computed by

$$l_f = \pm\frac{2\pi}{2(\tau_1-\tau_0)}; \qquad\qquad s = \frac{\Omega_{\tau_1}-\Omega_{\tau_0}}{\tau_1-\tau_0} \ .$$

We agree that the negative sign in the expression of l_f is chosen for clockwise librations.

□

REMARK 18. In the asymmetric libration case the procedure to terminate the numerical integration and to compute the proper frequencies would have to be different, but this case did not occur in the proper elements catalogs we have computed so far.

The following question arise in a natural way:

How different is the averaged solution, obtained with this method, from each individual non-averaged orbits which would be obtained by sampling the initial phases of the asteroid and the planets along their osculating orbits?

The answer to this question is the subject of the next chapter, anyway we draw here some immediate considerations.

The averaged solutions differ from whatever solution of the full equations by the short periodic perturbations. The latter can be significant as far as the orbital elements are concerned, especially if very close approaches take place.

The accuracy of the proper elements that we expect in the case of NEAs is much smaller than the accuracy in the case of Main Belt Asteroids (the latter is 10^{-3} over 10^7 yrs, see [38]); this is due to the strong perturbations they are subjected to, when their orbit gets close to the orbit of a planet. We have also to take into account that these perturbations make a NEA lifetime shorter than the lifetime of a Main Belt Asteroid.

7.3.3 Instability of the Athen-Apollo-Amor classification

The classification in Athen, Apollo and Amor is generally not stable for time spans of $100, 1\,000$ years or more [50] and the proper elements can be used to find the NEAs for which this classification is not valid for such times.

Over a longer time span, conditions of the form

$$a(1-e) < a' < a(1+e)$$

cannot assure the existence of crossings, because the assumption that the argument of perihelion ω precesses without changes in eccentricity and inclination is simplistic and erroneous.

Therefore the *Apollos* + *Athens* class, defined on the basis of the current osculating elements, is *not* equivalent to the class of Earth crossing asteroids, that are by definition the asteroids whose orbit crosses the orbit of the Earth during their secular evolution. Of course an Amor has to become later an Apollo to undergo a node crossing with the Earth.

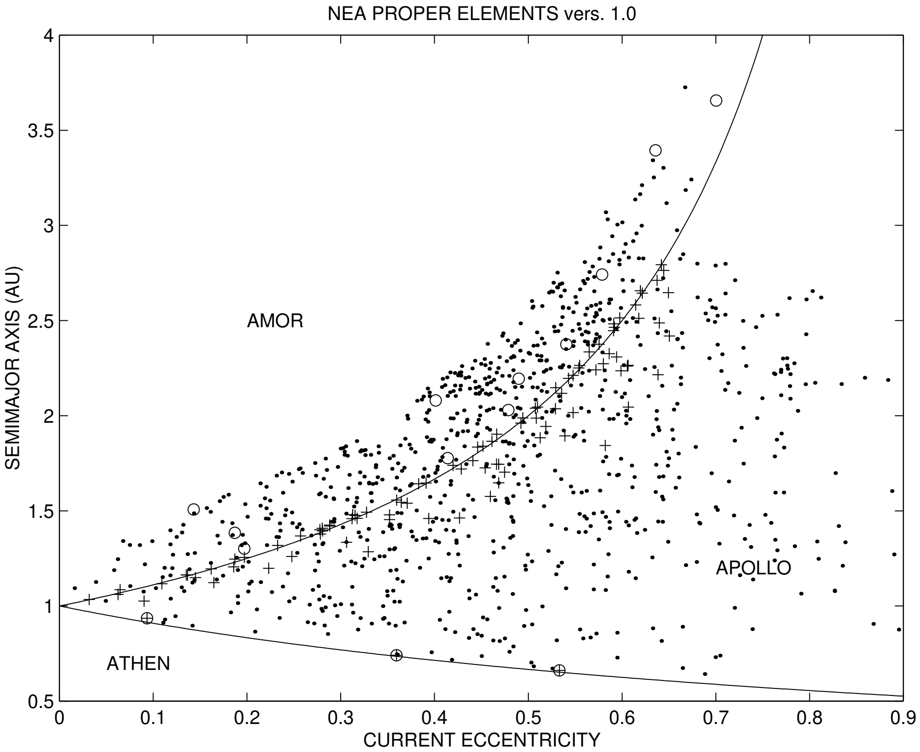

Figure 7.4: In the phase space (current e, a) we have plotted 987 asteroids drawing with crosses the 94 not Earth crossing Apollos, with circles the 11 Earth crossing Amors and with circled crosses the 3 not Earth crossing Athens. The two curves plotted represent $a' = a(1-e)$ and $a' = a(1+e)$ so that the *Earth crossing area* is between them.

In Figure 7.4 we have plotted in the plane (current e, a) the orbital elements of the 987 objects examined and we have enhanced the ones that, during their time evolutions, do not behave like their initial classification in Apollos, Amors and Athens should suggest: indeed, we have found 94 Apollos (out of 504) which are not Earth crossing (crosses), 11 Amors (out of 408) which are Earth crossing (circles) and only 3 Athens (out of 75) not Earth crossing (circled crosses).

We can observe that in Figure 7.5 the *non Earth crossing Apollos* have moved outside the Earth crossing area in the plane (minimum e, a) while in Figure 7.6, drawing the components (maximum e, a), the *Earth crossing Amors* have moved inside the Earth crossing area.

Remark 19. In this Section we have set the boundary between Apollos and Amors at a perihelion distance of exactly 1 AU for consistency with the model employed, having the planets, and in particular the Earth, on circular orbits.

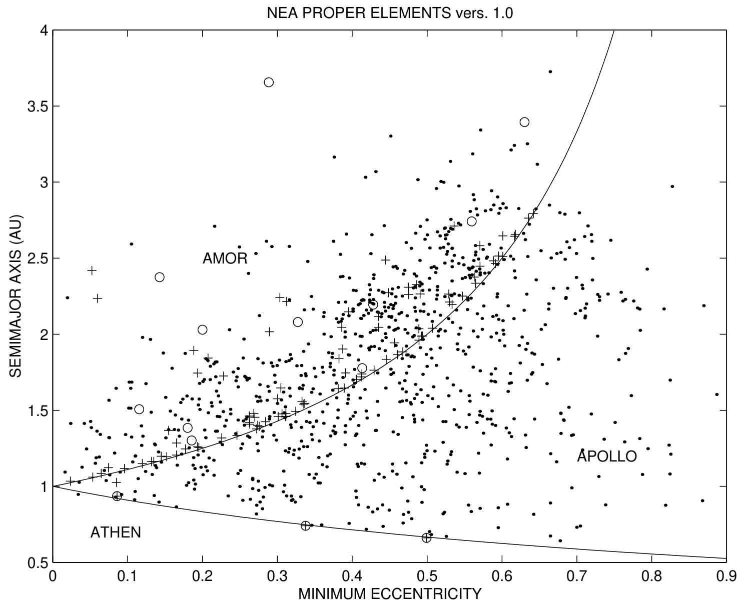

Figure 7.5: In the phase space (minimum e, a) some Apollo asteroids have moved in the non Earth crossing area; these *non Earth crossing Apollos* are marked with a cross.

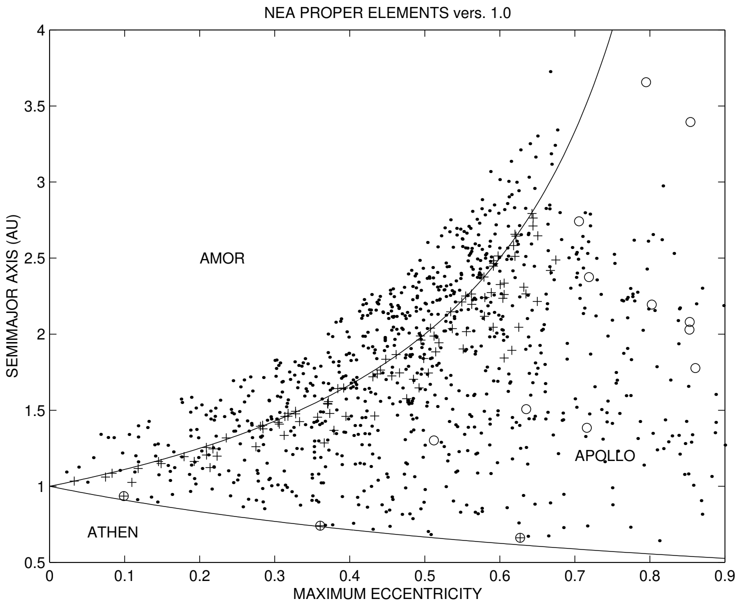

Figure 7.6: In the phase space (maximum e, a) 11 Amor asteroids have moved in the Earth crossing region; these *Earth crossing Amors* are marked with a circle.

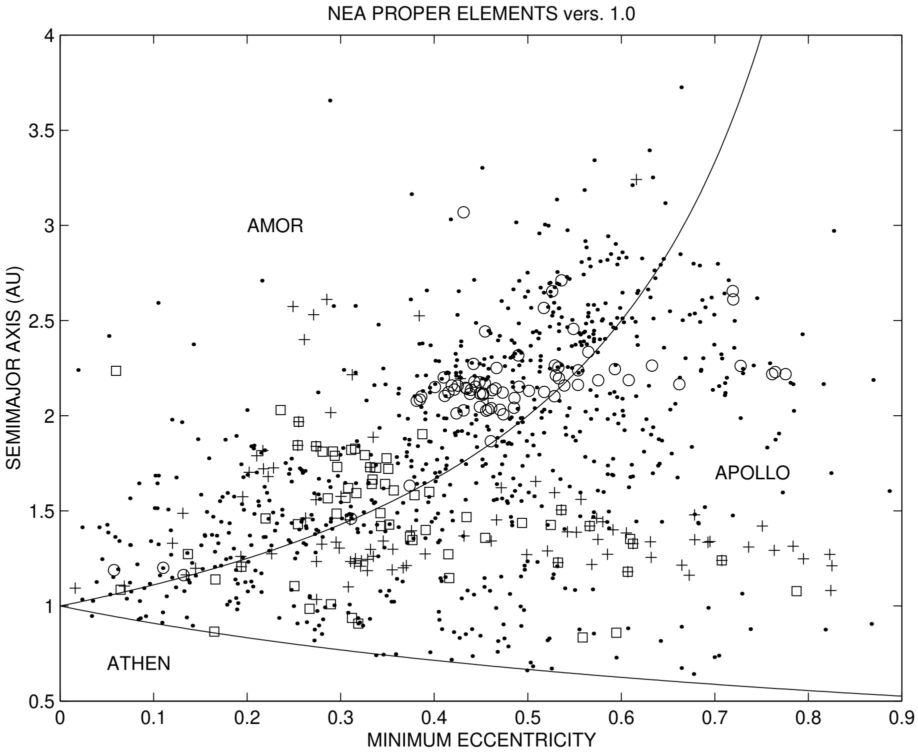

Figure 7.7: Map of the Near Earth Asteroids affected by one or more of the main secular resonances in the phase space (minimum e, a): the asteroids with $g \simeq g_6$ have been plotted with a circle, the ones with $s \simeq s_6$ with a square and the ones with $g \simeq g_5$ with a cross.

7.4 Secular resonances

From the computation of the proper frequencies it has been possible to draw a map of the dynamical structure of the Near Earth Asteroids, in particular identifying the location of the most important secular resonances.

In the plane (minimum e, a) we have plotted 987 NEA; we have marked with a circle the ones which we expect would be affected by the resonance ν_6 (that is $g - g_6 \simeq 0$), with a square the ones by the resonance ν_{16}, (that is $s - s_6 \simeq 0$), and with a cross the ones by the resonance ν_5 (that is $g - g_5 \simeq 0$). We have used a simple criterion, marking the ones with the secular divisor associated to the resonance smaller than 2 arcsec per year.

The well known transport mechanism towards high eccentricities due to the ν_6 resonance is clearly visible in Figure 7.7. It had already been studied by a secular theory and by numerical integrations in the main belt [59], [21]. Semianalytic theories confirmed its existence also near the Earth crossing region [47], [48], which could not be studied by the same theory because of the singularities appearing in the perturbing function.

Also the region of the Hungaria can be noticed: it is characterized by the presence of the ν_{16} resonance.

In addition to these already known dynamical structures determined by secular resonance, a new one appears from this plot: looking at the objects marked with crosses, also the ν_5 resonance seems to play a role in the increase of the eccentricity, mostly inside the Earth crossing region. This is consistent with the identification of the same resonance for orbits near the Earth but not crossing, as obtained by a semianalytic theory [48]. We are not aware of a numerical confirmation of this resonant effect for the orbits of NEAs, however we have to say that in [23] this resonance was detected in a simulation of the transport of meteorites from Mars to the Earth.

A confirmation of this fact would be useful, and even more useful would be an assessment of the possible role of the ν_5 resonance in injecting already Earth-crossing objects in orbits impacting the Sun.

Chapter 8

Reliability of the secular evolution theory

8.1 A test of reliability

In Chapters 4, 5 we have defined weak solutions of the averaged equation of motion (4.3), but there is no proof that these solutions will remain close for a long time span to the solutions of Hamilton's full equations.

In this chapter we shall present the results of a numerical test on the reliability of the weak averaged solutions and of the proper elements for NEAs obtained by the generalized averaging principle. We consider a sample of orbits and compare the results obtained by the solutions of the averaged equations with their corresponding values obtained by pure numerical integrations.

As the averaging principle for planet crossing orbits corresponds to the classical averaging when there is no crossing with any planet, we guess that such a comparison can result satisfying only when there are neither close approaches nor mean motion resonances with the planets, that are the applicability conditions of the classical averaging.

We have selected a sample from the whole set of NEAs in order to examine a large enough variety of different dynamical behaviors, including the ones for which we expect that the theory fails; the orbits of these NEAs are characterized by the following features:

1) ω – libration
2) ω – circulation
3) double node crossings
4) presence of mean motion resonances
5) presence of close approaches

For the same sample of NEAs, we have also performed numerical integrations taking into account the eccentricities and inclinations of the planets in the model, and we shall show also our numerical results also in this more realistic framework.

The results of this comparison are satisfying: the cases in which the theory fails are generally the same in which the classical averaging does not apply: mean motion resonances and close approaches with one or more planets, that could change the value of the semimajor axis, which is assumed to be constant in the secular (evolution) theory.

Some of the averaged quantities, like proper elements, are representative of the real behavior even if compared with the more realistic model, including the eccentricities and the inclinations of the orbits of the planets, but the crossing times between the orbits are not reliable at all if we do not take into account a more precise orbital configuration.

We notice that the computation of the crossing times is useful for several applications and, in particular, to study the possibility of collisions. Furthermore, if the integration time span is long enough, the real evolution gets away from the secular evolution computed in the circular planetary frame: for example the maximal value of the inclination for (433) Eros computed integrating the full equations in the elliptic inclined frame for a time span of 10^5 yrs is 15.27°, against the value of 10.82° presented in Table 8.3. Thus the need of a generalization of the averaging principle to elliptic and inclined planet orbits is evident, and this will be the subject of Chapter 9.

8.2 Numerical integrations

Numerical integrations play in Celestial Mechanics the same role as laboratory experiments do in other fields of sciences. This because no assumption or approximation is made to integrate the motion of the considered bodies and all the gravitational mechanisms (such as *resonances*) are taken into account in a natural way.

Nevertheless, it is well known that the orbits of planet crossing objects are strongly chaotic, with Lyapounov divergence times t_L of the order of 10^2 years (see [73], [81]); this is due to the sensible dependence on the initial orbital elements when there are planetary close encounters. The chaotic nature of the orbits implies that the results of numerical integrations cannot be seen as deterministic predictions of the evolution of the real objects, at least over times much longer than the Lyapounov time t_L.

Actually planet crossing orbits are very sensitive not only to the chosen initial conditions, but also to the assumed physical model of the Solar System, to the integration algorithm and to the rounding–off features of the computer used to perform the integrations. Introducing small changes to any of these options implies that after a time span of several times t_L, the computed orbits completely *loose memory* of their common origins. They are just possible evolutions, and are generally used to identify the most important dynamical mechanisms which may affect the real orbit, or to make statistical estimates on the object's lifetime, encounter rate and so on.

The computation of proper elements and frequencies requires to integrate the orbital motion of a NEA over a time span of the order of several 10^4 years, corresponding to at least half a precession period of the argument of perihelion ω, and over such a long time span we cannot expect to obtain exactly the same results from averaged and numerical integrations.

However a comparison is still possible and relevant if the considered body does not undergo deep close planetary encounters during the integration time span, and if it is not in a mean motion resonance with some planet.

To integrate numerically the equations of motion, it is necessary to use an accurate method since close approaches with the planets can cause fast variations of the orbital elements of NEAs. However, whereas a small time step is required when the integrated object is close to a planet (or the Sun) and moves very fast relative to it, a much longer time step can be used when the NEA is far from the planets. Therefore, an integration method using a variable step-size turns out to be the optimal choice. We have thus adopted the Bulirsch–Stoer extrapolation method [69] which, through a controlled variable step-size, allows to handle close approaches with planets much more accurately than with a fixed step-size.

The orbital motions of the considered bodies have been integrated in two models of the Solar System. The first one is very similar to the one used by the secular theory: it includes the planets from Venus to Saturn, initially placed on circular coplanar orbits. Note that our N-body numerical method consists in integrating the orbital motions of both the small bodies and the planets. So, the orbits of the planets are not forced to remain strictly circular and coplanar during their evolutions, which is the case in the secular theory. However, in the considered time spans, we have checked that the planetary inclinations remain close to zero and their eccentricities remain much smaller than if they started initially on their real orbits. We will thus consider the results of integrations in this model as the most direct comparison with those obtained by the secular theory. The second model of the Solar System includes the same planets now placed already initially on their inclined and elliptic orbits; this allows to check if the theory, though restricted to a computation in a circular planetary system, still gives relevant results for a comparison with numerical integrations in a more realistic planetary system.

The first problem that appears in doing this comparison is due to the fact that the initial positions of the asteroid and of the planets on their osculating orbits are relevant in the pure numerical integrations while they are not needed at all in the initial conditions of the averaged problem. Our integration method of the small bodies must take this into account.

Since the secular theory is constructed through the averaging over the fast angles corresponding to the mean anomalies, the computation by numerical integration of the average evolution and of proper elements and frequencies of a NEA orbit requires to perform several integrations of the same orbit over a time span covering at least half a cycle of the argument of perihelion, but changing for each integration the initial value of the mean anomaly, in order to cover the interval between 0° and 360°.

The average evolution over the mean anomalies can thus be obtained: proper elements and proper frequencies of the orbital angles are just the means of the values computed from each integration. Therefore, we have taken, for each considered body, the initial conditions of its orbital elements given at a certain epoch (see Table 8.1), and performed four numerical integrations, considering as initial mean anomaly 0°, 90°, 180° and 270°,

Body	a (AU)	ω (°)	Ω (°)	e	I (°)
(433) Eros	1.45823	178.640	304.411	0.222863	10.829
(1036) Ganymed	2.65864	132.325	215.787	0.538201	26.645
(1981) Midas	1.77611	267.720	357.097	0.650113	39.831
(5496)	2.43361	118.204	101.101	0.637850	68.018
1991 TT	1.19514	217.843	192.461	0.161822	14.836
1992 YD_3	1.16267	173.475	274.330	0.134809	26.912
1998 SF_{35}	1.68520	253.702	218.699	0.273587	35.195
1999 AN_{10}	1.45870	268.293	314.508	0.562172	39.931
Geminids	1.35000	324.800	260.300	0.896000	23.600

Table 8.1: Initial orbital elements of the NEAs used for the comparison between the secular theory and numerical integrations. The NEAs elements are referred to the ecliptic J2000 and are given at epoch JD 51800. The name Geminids refers to the *Geminids meteor stream*, whose considered mean elements are the ones given in [20].

successively. Each integration has been performed over a time span allowing to determine the precession/libration period of the argument of perihelion. We have then computed the precession/libration frequencies of the angles ω, Ω, l_f described in Chapter 7 for each of the four evolutions. The average frequency is just the mean of these four quantities.

We have also checked the differences in the evolution between each of the four clones and their mean value; this has been done in order to check that the mean values are suitable to be used in the comparison. In most cases the mean is close to each of the four evolutions. In Table 8.6 we discuss a case for which one of the clones differs from the other ones due to close approaches: this modifies its evolution in a strong way.

8.3 Results of the comparison

We show the result of the comparison for a sample of NEAs, whose names and initial conditions are written in Tables 8.1 and 8.9.

In Table 8.1 we put the data of the objects for which this comparison is expected to be successful, while in Table 8.9 we put some examples of NEAs for which the hypotheses for the applicability of the classical averaging do not hold, and in these cases we expect that the weak averaged solutions are not representative of the real solutions.

In Table 8.1 we have also inserted the data of the mean orbit of the *Geminids meteor stream* (see [20]) to have information on the applicability of the secular theory to orbits with high eccentricity.

8.3.1 Results for proper elements

The output of the two methods is given at different values of the time, so that, in order to do a consistent comparison, we have to interpolate these values to compute a good approximation of the elements obtained with both methods at any time. The interpolation is done by means of MATLAB software package, which is used also to produce all the figures in this chapter.

As in general the evolution of the elements in the secular theory is represented by piecewise differentiable curves, we interpolate the output of the secular evolution algorithm with piecewise polynomials of degree 4 in each time interval between two consecutive node crossings.

On the other hand the numerical integrations output is a discretization of a smooth solution, except for initial conditions that result in a collision (and not simply in a node crossing): we choose to interpolate these data (using only the ones in the time span of the secular integration) by an 8th degree polynomial, whose coefficients are computed by taking into account only the values of the output in the time range of the secular evolution.

We follow a different method for the interpolation of the perihelion argument of the three ω-librators (1981), (5496) and 1999 AN_{10}: for each of them, we interpolate the numerical output with a 15th degree polynomial taking into account the whole time span of the numerical integration (50000 yrs) in order to get both a maximal and a minimal value over a whole cycle of the librating angle. In the secular theory we perform a part of this computation by symmetry, that is, if we have got the maximal value of ω we can compute the corresponding minimum using the mirror symmetry in the level curves of the averaged Hamiltonian.

In Tables 8.2, 8.3 and 8.4 we present the set of proper elements obtained in this way for the objects listed in Table 8.1, whose initial conditions are suitable to apply the secular theory. The agreement of the elements is very good: it is of the order of 5×10^{-3} both for the eccentricity and the inclination (in radians) over a time span of the order of 10^4 yrs. The time span used in the secular computation, and in this check, is the time needed to compute proper elements and frequencies employing the symmetries in the level curves of the averaged perturbing function: for librators it is the time needed to do more than half and less then one cycle of the libration argument lf; for circulators it is the time needed to do more than a quarter and less than half a cycle of ω.

In these tables we also present the set of the elements obtained by integrating the full equations including the eccentricity and the inclination of the planets: as one can see, in most cases also these elements are again close to the previous ones.

We have selected two of these asteroids to present a graphic comparison of the time evolution of their orbital elements and of their secular evolution in the plane (ω, e) computed with the secular theory and with numerical integrations in the same framework, that is with the planets on circular coplanar orbits (see Figures 8.1, 8.2). The evolutions of the elements computed with the two methods almost overlap in each plot corresponding to these two examples.

We have computed also the maximal difference between the interpolated curves in the time span of the secular evolution output, examining their values with a time step of 1 yr; these values are written at the top of each evolution diagram of the orbital elements presented here.

	e_{MIN}			e_{MAX}		
Body	ST	NI circular	NI elliptic	ST	NI circular	NI elliptic
(433)	0.22285	0.22272	0.22251	0.23307	0.23289	0.22802
(1036)	0.39158	0.39467	0.38484	0.59311	0.59394	0.58083
(1981)	0.34872	0.34978	0.39586	0.65075	0.65029	0.65011
(5496)	0.51232	0.51184	0.51231	0.92776	0.92896	0.92774
1991 TT	0.15221	0.15462	0.15210	0.17910	0.17506	0.17564
1992 YD_3	0.13433	0.13435	0.13400	0.19174	0.19569	0.19399
1998 SF_{35}	0.10162	0.10264	0.11541	0.30994	0.31019	0.32266
1999 AN_{10}	0.28973	0.28732	0.35928	0.56566	0.56188	0.56346
Geminids	0.82669	0.82687	0.81177	0.90775	0.90777	0.91109

Table 8.2: Maximal and minimal value of the eccentricity for each object in Table 8.1; the columns labeled with ST are the results of the Secular Theory while the ones with NI are the results of the Numerical Integrations. The agreement of the elements computed in the same framework (see columns with label *circular*) with both methods is very good. Also, in most cases the elements computed with NI taking into account the eccentricity and the inclination of the planets (columns with label *elliptic*) are quite close to the ST proper elements.

	I_{MIN} (°)			I_{MAX} (°)		
Body	ST	NI circular	NI elliptic	ST	NI circular	NI elliptic
(433)	10.06653	10.06472	10.14120	10.82960	10.83039	10.82938
(1036)	20.66594	20.67313	24.43023	35.04730	35.05927	35.85769
(1981)	39.78069	39.79393	39.77657	51.49418	51.43053	49.64942
(5496)	39.40863	39.16214	39.56608	70.38552	70.37499	70.44334
1991 TT	14.16372	14.33836	14.73610	15.16664	15.04331	15.69457
1992 YD_3	25.80634	25.71217	26.62592	26.91905	26.91729	27.02954
1998 SF_{35}	34.23954	34.23136	34.76566	37.81302	37.79242	39.34574
1999 AN_{10}	39.92535	39.94773	39.72519	48.50540	48.55123	49.04121
Geminids	14.06904	14.02986	16.03717	43.69797	43.70156	45.97490

Table 8.3: Maximal and minimal value of the inclination (in degrees) for each object of Table 8.1.

	ω_{MIN} (°)			ω_{MAX} (°)		
Body	ST	NI circular	NI elliptic	ST	NI circular	NI elliptic
(1981)	248.61698	248.51999	251.2483	291.38302	291.56100	288.5705
(5496)	53.46308	52.59263	52.2180	127.40737	128.11562	128.6610
1999 AN_{10}	251.29747	251.23062	256.8707	288.70253	288.73011	286.4692

Table 8.4: Maximal and minimal value of the perihelion argument (in degrees) for the ω–librators considered in Table 8.1. The agreement between secular theory and numerical integrations in the same framework for (5496) is not as good as for the other ones: this error is introduced by the interpolation in fact, in this case, even a 15th degree polynomial cannot approximate the data in a suitable way.

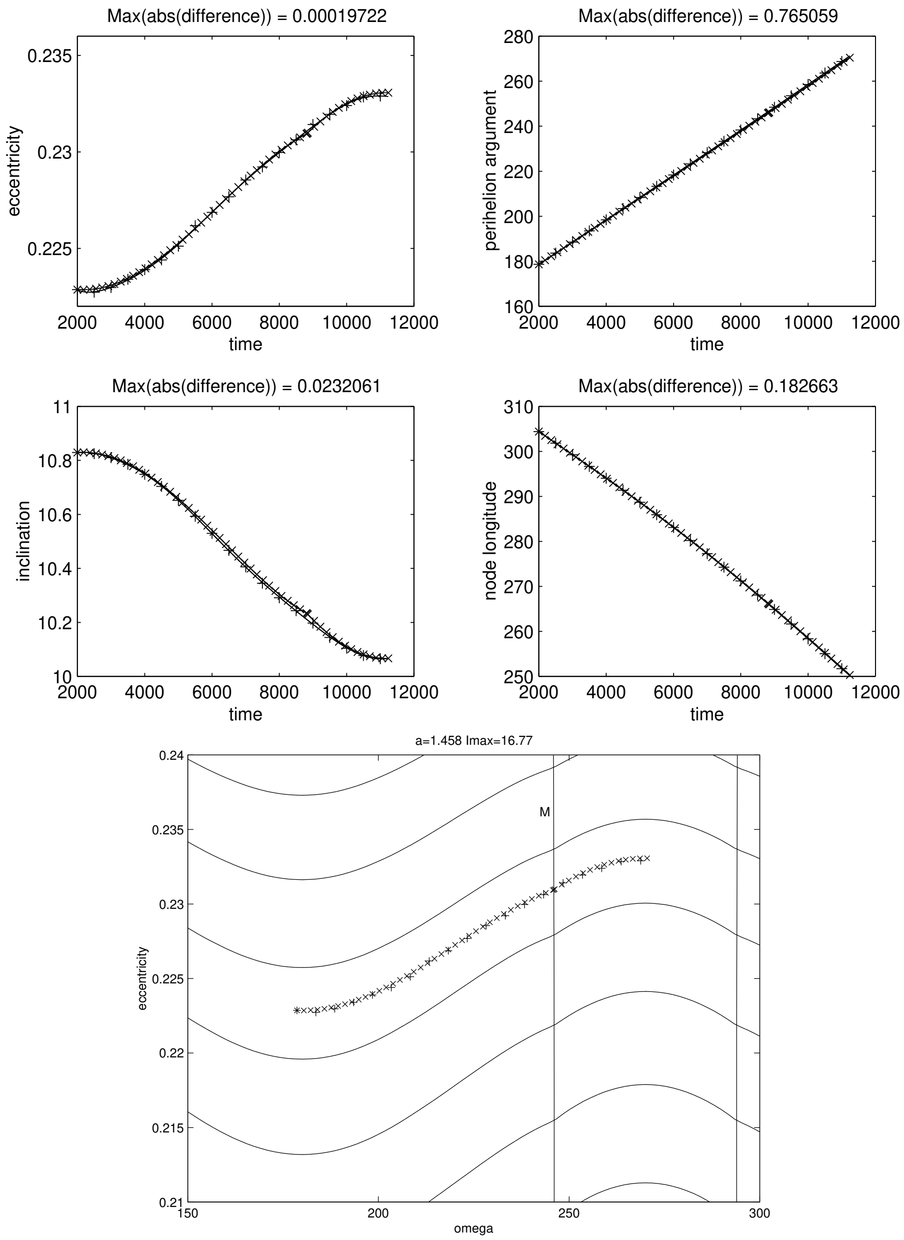

Figure 8.1: The four diagrams in this figure represent the time evolution comparison of the orbital elements e, ω, I, Ω of the asteroid (433) Eros. The output of the secular theory is plotted on each of these diagrams with the `x` characters, while we use $+$ for the output of the numerical integration in the circular coplanar case; in the same diagrams we plot also the interpolation of both sets of data. The diagram at the bottom represents the comparison of the evolutions in the plane (ω, e): the background lines are the level curves of the averaged Hamiltonian and the node crossing curves with Mars (M). On the top of this figure we put some data concerning the secular evolution: they are the constant value a of the semimajor axis and the maximal value of the inclination *Imax*. The agreement between the secular theory and numerical integrations is evident: the two evolutions almost overlap.

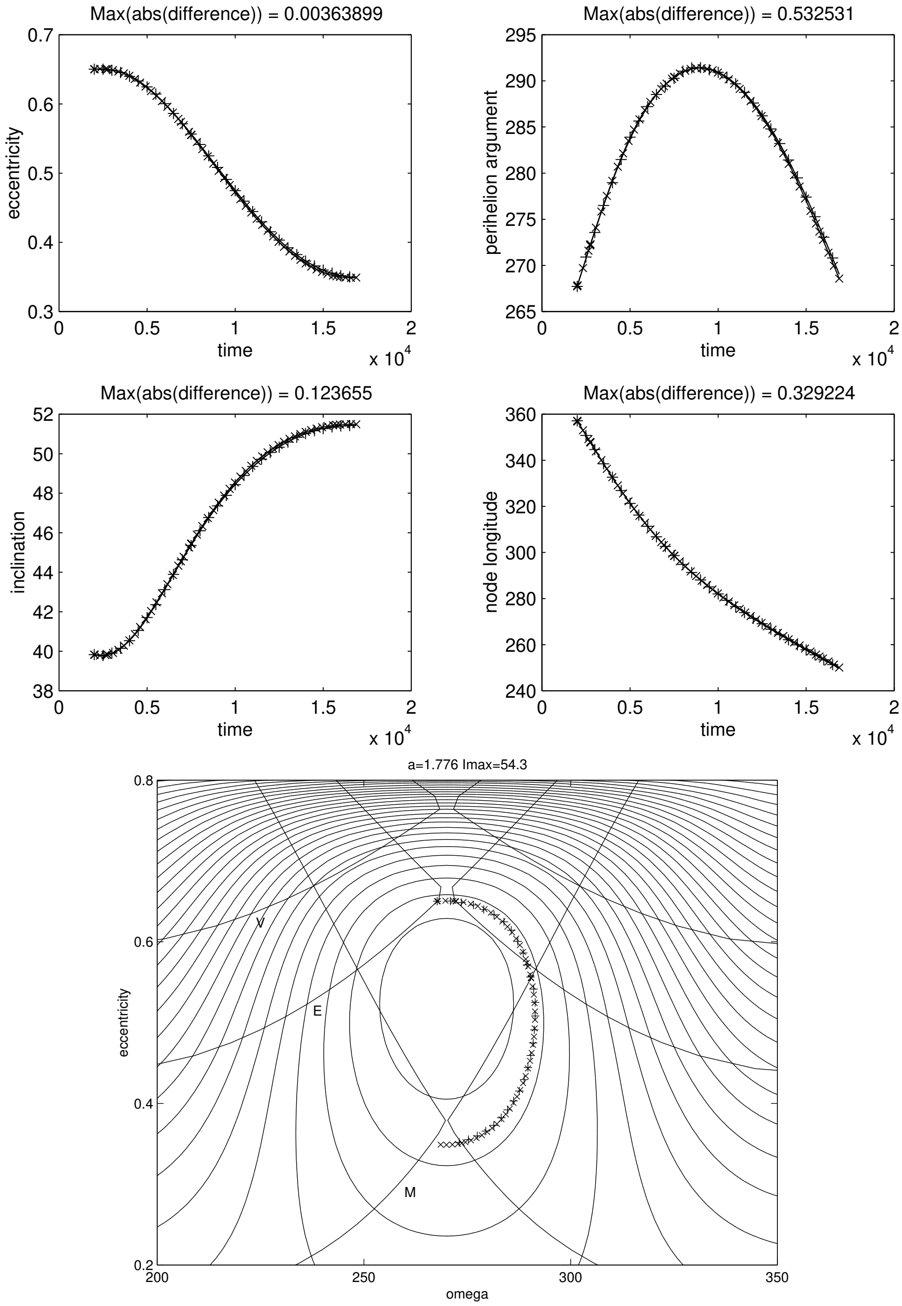

Figure 8.2: Same as Figure 8.1 for the asteroid (1981) Midas: this example shows the agreement of the data also in the ω–libration case. Node crossing lines with Venus (V), the Earth (E) and Mars (M) are plotted in the plane (ω, e).

	g ("/year)			s ("/year)		
Body	ST	NI circular	NI elliptic	ST	NI circular	NI elliptic
(433)	14.675	14.876	15.105	−21.101	−21.136	−20.022
(1036)	14.461	15.854	13.778	−74.663	−76.738	−67.512
1991 TT	10.713	8.522	9.736	−19.832	−19.530	−19.704
1992 YD_3	4.352	5.414	4.422	−13.459	−13.635	−12.903
1998 SF_{35}	−2.241	−2.169	−1.870	−16.565	−16.565	−16.057
Geminids	3.405	3.404	3.375	−36.276	−36.358	−37.109

Table 8.5: Proper frequencies of the orbital longitude of perihelion (g) and nodal longitude (s) for the ω–circulator objects in our sample. The values presented here have been computed by means of the secular theory (ST) and numerical integrations (NI) of the orbits in the system including the planets from Venus to Saturn on circular orbits, and in the framework including the same planets on their elliptic inclined orbits.

	s ("/year)			lf ("/year)		
Body	ST	NI circular	NI elliptic	ST	NI circular	NI elliptic
(1981) Midas	−25.625	−25.650	−26.632	−45.672	−45.422	−50.573
(5496)	−44.503	−45.052	−44.482	−87.383	−88.901	−87.807
1999 AN_{10}	−17.788	−17.763	−18.962	−27.755	−27.377	−30.137

Table 8.6: Proper frequencies for ω–librator objects. Note that the comparison of the frequencies for (5496) in the same framework is not as good as for the other ones: this is due to a close approach in one of the four numerically computed evolutions that we selected. If we discard this particular evolution to perform the average of the numerical integrations in the circular coplanar case, we obtain $s = -44.725$ and $lf = -87.881$.

For each object of Table 8.1 we obtain an accuracy of $\approx 10^{-2}$ for the evolution of the eccentricity and $\approx 6 \times 10^{-3}$ for the evolution of the inclination (if considered in radians) over a time span of the order of 10^4 yrs.

8.3.2 Results for proper frequencies

In the computation of the proper frequencies we also have to distinguish between the ω–circulators (Table 8.5) and the ω–librators (Table 8.6): the frequencies for the circulators are the longitude of perihelion g and the longitude of node s, while for the librators they are the libration frequency lf and the longitude of node s. Actually we can observe that the frequency of the perihelion argument is zero for the ω–librators, hence g is equal to s in this case.

As one can see, the agreement of the secular theory with numerical integrations in both regimes is fairly good, even if there are cases (like (1036) Ganymed, 1991 TT, 1992 YD_3) for which there are discrepancies of more than 1 $arcsec/yr$, due to the presence of

	e_{MIN}		e_{MAX}		I_{MIN}		I_{MAX}	
Body	ST	NI	ST	NI	ST	NI	ST	NI
(433)	0.2228	0.2227	0.2335	0.2331	10.0313	10.0325	10.8299	10.8289
1998 SF_{35}	0.0964	0.0967	0.3138	0.3136	34.1240	34.1318	37.8441	37.8413

Table 8.7: Maximal and minimal values of the eccentricity and inclination (in degrees) for (433) Eros and 1998 SF_{35}, computed in a planetary frame including only Jupiter and Saturn on circular coplanar orbits.

	g (”/year)		s (”/year)	
Body	ST	NI	ST	NI
(433)	12.712	12.721	-15.663	-15.648
1998 SF_{35}	-1.970	-1.930	-15.767	-15.759

Table 8.8: Proper frequencies for (433) Eros and 1998 SF_{35} computed in a planetary frame including only Jupiter and Saturn on circular coplanar orbits.

close approaches in the numerical evolution of some of the four clones. These encounters are not deep enough to change in a sensible way the value of the proper elements with respect to the ones computed semianalytically, while the changes in the frequencies are more evident.

Also in this case most of the frequencies do not change drastically when we compute them with the planets on their elliptic inclined orbits.

We conclude this paragraph with the following discussion: as we have seen, including the inner planets and taking into account the node crossings in the secular theory increases much the difficulty of the problem, it is then interesting to analyze whether removing the inner planets from the system would affect the computation of the proper elements and frequencies for NEAs. This can tell us how the planetary crossing effect contributes for these asteroids and whether considering the inner planets to compute their proper elements and frequencies is really justified. We notice that a first analysis of the importance of the inner planets in the computation of the proper elements and frequencies is in [54].

We have chosen two bodies from our sample: (433) Eros and 1998 SF_{35}. Using a planetary system including now only Jupiter and Saturn on circular coplanar orbits, we have recomputed the values of their proper elements and frequencies by the semianalytic method and by numerical integrations, as explained previously: Tables 8.7, 8.8 show the results. Though the agreement with the values of the proper elements computed in the model including the inner planets remains satisfactory, it is not the case at all for the proper frequencies.

We have thus a confirmation of the importance of including the inner planets in the planetary system, despite the greater difficulty introduced by the presence of node crossings.

8.3.3 Limits of the validity of the theory

In this section we study the cases for which the secular theory does not give good results. The results are wrong when there is a mean motion resonance between the asteroid and a planet, or when there is a close approach that causes a significant change of the semimajor axis.

Here we shall compare the output of the secular theory and numerical integrations in the same framework only.

In Table 8.10 we show the proper elements of the objects of Table 8.9, while in Table 8.11 we present their frequencies.

Body	a (AU)	ω (°)	Ω (°)	e	I (°)
(887) Alinda	2.48408	350.050	110.709	0.563456	9.310
(1685) Toro	1.36718	126.973	274.403	0.435949	9.378
(4179) Toutatis	2.51030	274.689	128.376	0.634349	0.470
2000 CO_{101}	1.07529	65.138	353.182	0.089574	15.327

Table 8.9: Initial orbital elements (referred to the ecliptic J2000 and given at epoch JD51800) of the asteroids for which the secular theory fails in our numerical experiments. Three cases of failure are due to a mean motion resonance ((887) Alinda, (1685) Toro, (4179) Toutatis), while the other one is due to close approaches (2000 CO_{101}).

	e_{MIN}		e_{MAX}		I_{MIN}		I_{MAX}	
Body	ST	NI	ST	NI	ST	NI	ST	NI
(887) Alinda *	0.5623	0.5410	0.5742	0.5766	5.2148	7.3940	9.6125	9.5120
(1685) Toro	0.4307	0.4310	0.4371	0.4377	9.1568	9.1729	10.3004	10.2944
(4179) Toutatis *	0.6342	0.6201	0.6343	0.6531	0.4684	0.8093	0.9103	0.9703
2000 CO_{101}	0.0754	0.0795	0.1456	0.1376	13.8591	13.9053	15.5643	15.4236

Table 8.10: Proper eccentricity and inclination (in degrees) for the asteroids of Table 8.9, whose evolution computed with the secular theory (ST) is not reliable due to close approaches with planets (2000 CO_{101}) or to mean motion resonances ((887),(1685) and (4179)). The asterisks tells us when the interpolation of the numerical output is not reliable, so that in the NI columns we simply write the maximal/minimal values of the numerical output over a time span of the order of the secular computation time.

In these examples we can observe the presence of mean motion resonances with different planets and discuss their effects in the computation of proper elements. For example the asteroids (4179) and (887) are in mean motion resonance with Jupiter: this cause a drastic difference in their evolution computed with both methods, overall in the computation of the frequencies. The case of (1685) is different: it becomes resonant with the Earth when it gets close to it; this is a protection mechanism that has already been studied in [53]. For this kind of resonances the secular theory still gives quite reliable results.

In Figures 8.4, 8.5 we show the comparison of the evolutions computed with the two methods for (1685) Toro and for 2000 CO_{101}. We also show the variation in semimajor axis for 2000 CO_{101} in Figure 8.3: this is due to close approaches with the Earth.

	g ("/year)		s ("/year)	
Body	ST	NI	ST	NI
(887)	42.943	6.341	−120.532	−72.125
(1685)	8.418	8.655	−28.794	−28.837
(4179)	46.790	15.164	−211.607	−134.682
2000 CO_{101}	−10.970	−8.947	−22.362	−23.505

Table 8.11: Proper frequencies for the asteroids of Table 8.9.

To conclude this section we note that there is another, very rare, case of failure of the averaging method: when the asteroid orbit has a very low inclination, so that a local minimum of the distance function between the orbits can be far from a mutual node (see Chapter 10), then the Wetherill functions d, d_1, used in the semianalytic method, does not approximate the distance D at the correct points. In this case the averaged integration times can become quite long, as we have realized in the case of the asteroid 1998 *MZ* (that has an initial inclination of about $I = 0.135°$), whose evolution computed by the secular theory requires more than ten times a typical CPU time required for an averaged integration.

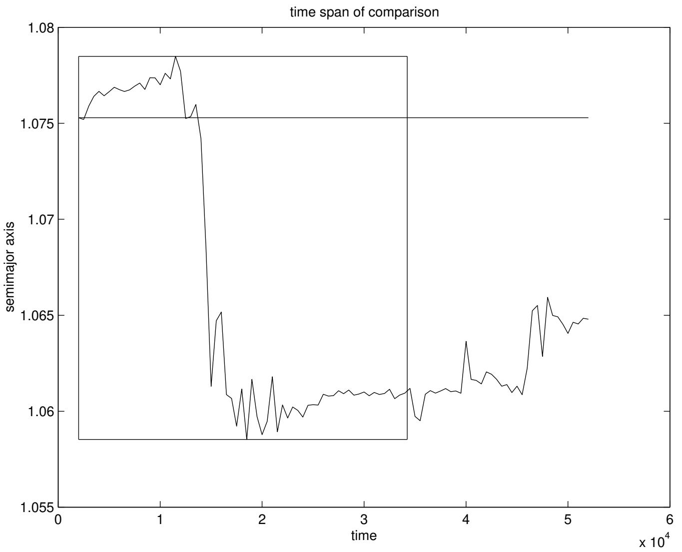

Figure 8.3: Changes in the semimajor axis of the asteroid 2000 CO_{101}, due to the close approaches with the Earth. The constant value $a = 1.07529$ AU, assumed in the secular theory, is plotted together with the values obtained by numerical integrations. The square delimits the time span used for the averaged computation.

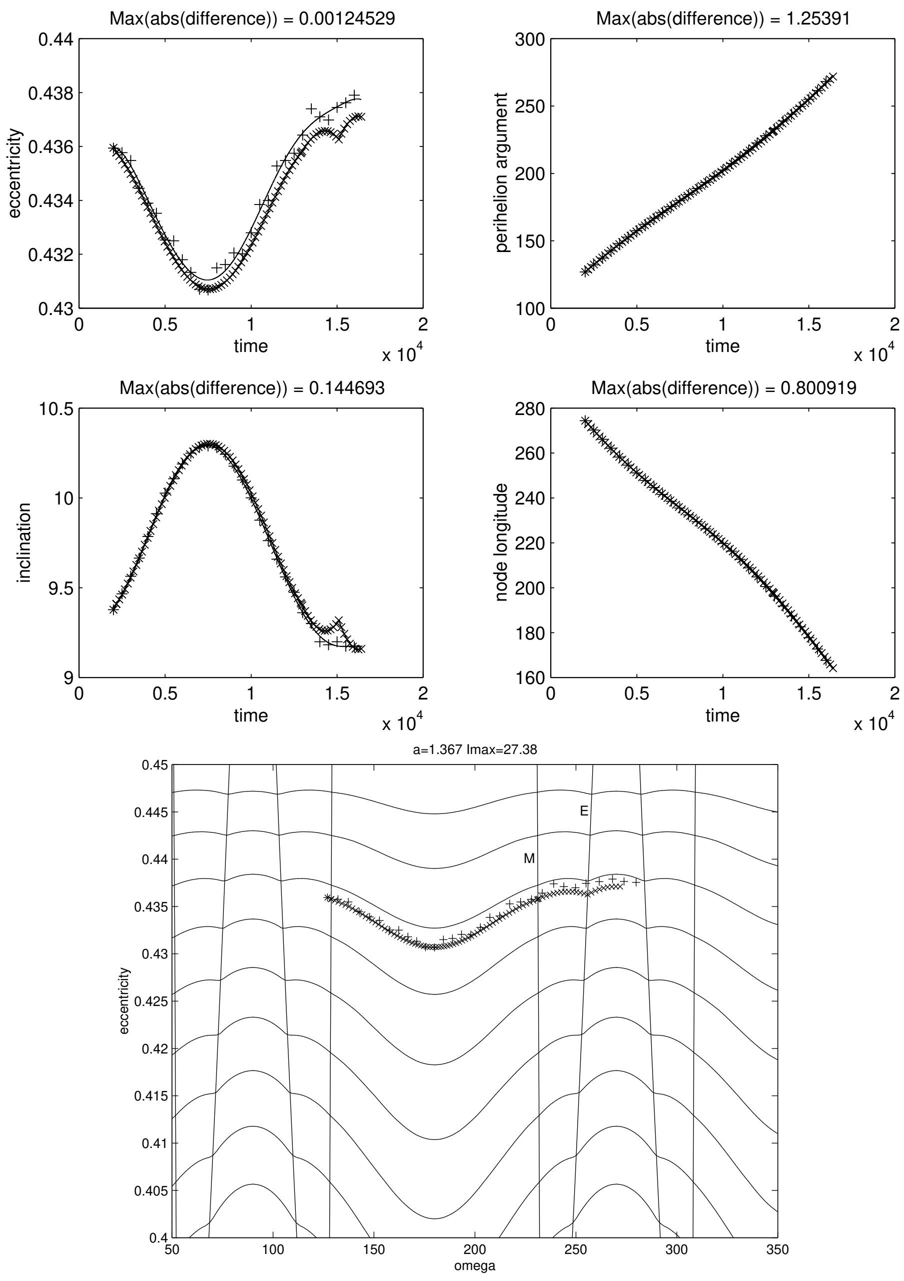

Figure 8.4: Same as Figure 8.1 for the asteroid (1685) Toro: in this case the mean motion resonance with the Earth plays a protective role (see [53]); proper elements and frequencies computed with the secular theory are still fairy good in this case. In this figure we see that the polynomial fit of the numerical integration data is not very accurate when this object crosses the Earth or Mars as the corners produced by the crossing planets perturbation in the level lines of the averaged Hamiltonian are quite evident.

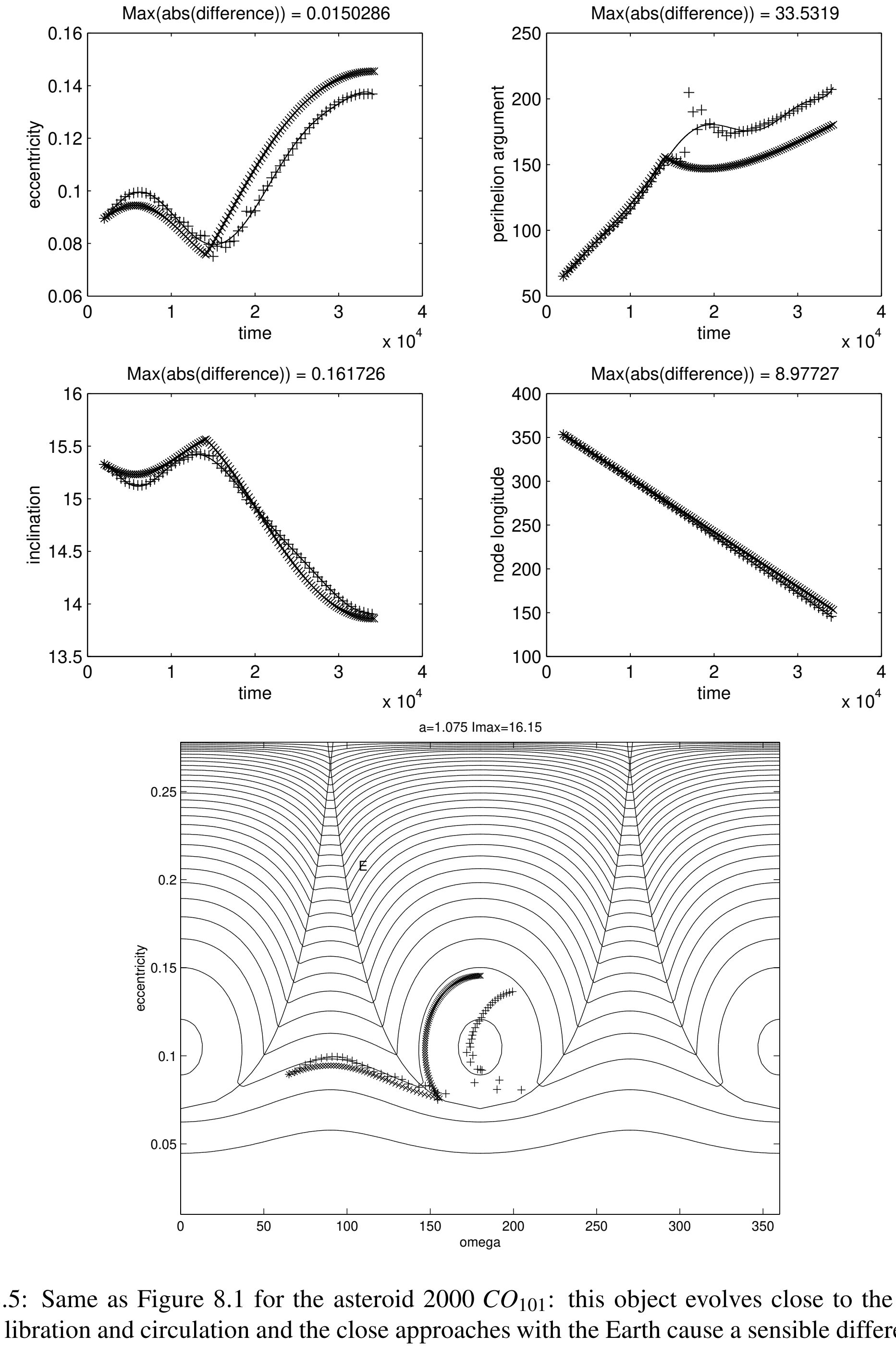

Figure 8.5: Same as Figure 8.1 for the asteroid 2000 CO_{101}: this object evolves close to the separatrix between libration and circulation and the close approaches with the Earth cause a sensible difference in the two evolutions of ω and in the frequencies (see Table 8.11). Nevertheless the difference between the values of e and I computed with the two methods remains small.

Chapter 9

Averaging on planet-crossing orbits: elliptic inclined case

9.1 Introduction

In this chapter we explain a generalization of the secular theory described in Chapters 4, 5, including the eccentricities and the inclinations of the planets in the model: this work is based on [26]. We shall make use of a formalism that allows to show the analogies with the circular coplanar case, so that most of the formulas are very close to the ones of Chapters 4 and 5 and we shall even use the same letters for some of them, but we have to keep in mind that most of these letters have a different meaning and they hide a different analytic expression.

9.2 The averaged equations

Let us consider a restricted three body problem: Sun, planet, asteroid. We use, in a heliocentric reference frame, the Delaunay variables (L,G,Z,ℓ,g,z) for the asteroid and (L',G',Z',ℓ',g',z') for the planet.

The orbit of the planet is completely determined: it is the solution of the two body problem Sun, planet. If we set $\mathcal{E}_{\mathcal{D}} = (G,Z,g,z)$, the averaged equations of motion for the asteroid can be written in the following form:

$$\dot{\widetilde{\mathcal{E}}}_{\mathcal{D}} = -\mathcal{J}\,\overline{\nabla_{\mathcal{E}_{\mathcal{D}}} R}^{\,t} \tag{9.1}$$

where $\widetilde{\mathcal{E}}_{\mathcal{D}} = (\widetilde{G},\widetilde{Z},\tilde{g},\tilde{z})$ are averaged Delaunay variables, $\mathcal{J}$ is the 4×4 matrix

$$\mathcal{J} = \begin{bmatrix} O_2 & -I_2 \\ I_2 & O_2 \end{bmatrix}$$

(O_2 and I_2 are respectively the 2×2 zero and identity matrices) and $\overline{\nabla_{\mathcal{E}_{\mathcal{D}}} R}^{\,t}$ is the transposed vector of the integral average over (ℓ,ℓ') of the partial derivatives of the perturbing function R with respect to $\mathcal{E}_{\mathcal{D}}$.

We consider the solutions of equations (9.1) as representative of the solutions of the full equations of motion if there are no mean motion resonances between the asteroid and the planet and if no close approaches occur between them.

9.3 Weak averaged solutions

If there are no crossings between the orbits, by the theorem of differentiation under the integral sign, the averaged equations of motion (9.1) are equivalent to Hamilton's equations with the averaged perturbing function $\overline{R}$:

$$\dot{\tilde{\mathcal{E}}}_{\mathcal{D}} = -\mathcal{J}\left[\nabla_{\mathcal{E}_{\mathcal{D}}}\overline{R}\right]^t \tag{9.2}$$

where

$$\overline{R} = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} R\, d\ell\, d\ell' \,. \tag{9.3}$$

We shall follow the same steps of Chapter 4: we shall prove that it is possible to define piecewise smooth solutions of equations (9.2) and that the loss of regularity corresponds exactly to the intersection configurations of the orbits. At the node crossing the right hand sides of equations (9.2) can be assigned two different values like in the circular coplanar case, corresponding to a passage to the limit coming from both sides of the node crossing hyper-surfaces, defined later.

We define as *weak averaged solutions* the solutions of (9.2). We observe that they correspond to the classical averaged solutions as far as their trajectories in the phase space do not pass through node crossing conditions.

In the next sections we shall see that it is possible to write $\overline{R}$ as a function of a set of variables, called *mutual elements*; these elements are almost everywhere regular functions of the Delaunay variables and are defined by the mutual position of the osculating orbits of the asteroid and the planet. Using the *chain rule* we shall see that we can write the equations of motion for the asteroid in the more explicit form

$$\begin{cases} \dot{\tilde{G}} = \dfrac{\partial\overline{R}}{\partial\mathcal{E}_{\mathcal{M}}}\dfrac{\partial\mathcal{E}_{\mathcal{M}}}{\partial g} \\ \dot{\tilde{Z}} = \dfrac{\partial\overline{R}}{\partial\mathcal{E}_{\mathcal{M}}}\dfrac{\partial\mathcal{E}_{\mathcal{M}}}{\partial z} \end{cases} \qquad \begin{cases} \dot{\tilde{g}} = -\dfrac{\partial\overline{R}}{\partial\mathcal{E}_{\mathcal{M}}}\dfrac{\partial\mathcal{E}_{\mathcal{M}}}{\partial G} \\ \dot{\tilde{z}} = -\dfrac{\partial\overline{R}}{\partial\mathcal{E}_{\mathcal{M}}}\dfrac{\partial\mathcal{E}_{\mathcal{M}}}{\partial Z} \end{cases} \tag{9.4}$$

where $\mathcal{E}_{\mathcal{M}}$ is a suitable subset of the mutual elements.

REMARK 20. In Chapter 4 the variable Z is an integral of the motion, corresponding to the third component of the angular momentum. Here this property is not valid any more.

The advantages coming from the use of the mutual elements will be clear in the next sections: they will allow us to use the same formalism as in the circular coplanar case. We shall skip the 'tilde' over the averaged variables in the following, to avoid the use of a heavy notation.

9.4 The mutual reference frame

Given two closed non-coplanar osculating orbits of an asteroid and a planet in the three body problem Sun, planet, asteroid, we recall that the *mutual nodal line* is the intersection between the two orbital planes. Furthermore we define as *mutual node* each pair of points on the mutual nodal line, one belonging to the orbit of the asteroid and the other to the one of the planet, that lie on the same side of the mutual nodal line with respect to the common focus of the two conics. In this framework there are two mutual nodes, the *ascending* and the *descending* one: they differ in the change of sign of the z component along the asteroid orbit (negative to positive in the first case and vice-versa in the second).

The *mutual reference frame* (x,y,z) is a heliocentric reference system such that the x-axis is along the mutual nodal line, directed towards the ascending mutual node; the y axis lies on the planet orbital plane, so that the orbit of the planet lies on the (x,y) plane (see Figure 9.1). We shall use the further convention that the positive z-axis is oriented as the angular momentum of the planet.

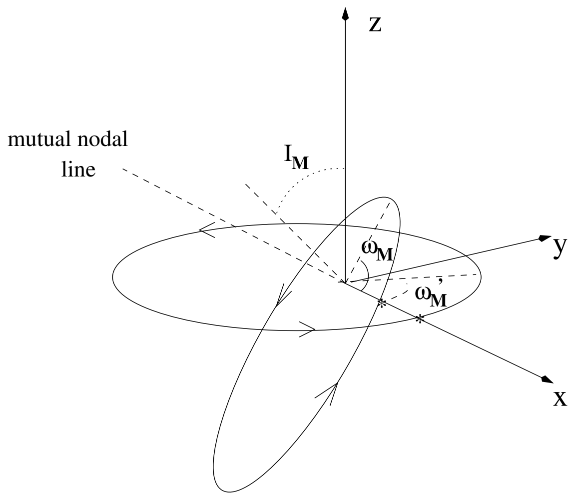

Figure 9.1: The *mutual reference frame*: the orbit of the planet lies on the (x,y) plane; the x axis is along the mutual nodal line and is oriented towards the ascending mutual node, marked with asterisks in the figure.

Let ω_M, ω'_M be the mutual pericenter arguments (the counter-clockwise angles between the x axis and the pericenters) of the orbit of the asteroid and of the planet respectively, and let I_M be the mutual inclination between the two conics. We define as *mutual elements* the set of variables $\{a, e, a', e', \omega_M, \omega'_M, I_M\}$.

We can express ω_M, ω'_M, I_M as functions of the Keplerian elements $\omega, \Omega, I, \omega', \Omega', I'$, defined using a fixed heliocentric reference frame. Let us consider the unit vectors $\mathcal{N}_{ast}$ and $\mathcal{N}_{pl}$, corresponding to the direction of the angular momentum of the asteroid and the planet; each of them is orthogonal to one of the osculating orbits and they are functions of the Keplerian elements:

$$\mathcal{N}_{ast} = \begin{pmatrix} \sin\Omega\sin I \\ -\cos\Omega\sin I \\ \cos I \end{pmatrix}; \qquad \mathcal{N}_{pl} = \begin{pmatrix} \sin\Omega'\sin I' \\ -\cos\Omega'\sin I' \\ \cos I' \end{pmatrix}.$$

The unit vector pointing to the ascending mutual node is orthogonal to both $\mathcal{N}_{ast}$ and $\mathcal{N}_{pl}$, so it is defined by

$$\mathcal{A}_{nod} = \frac{\mathcal{N}_{pl} \wedge \mathcal{N}_{ast}}{|\mathcal{N}_{pl} \wedge \mathcal{N}_{ast}|} \tag{9.5}$$

where $\wedge$ is the symbol of the vector product in $\mathbb{R}^3$. The mutual inclination $I_M \in [0,\pi[$ is defined by

$$\cos I_M = \langle \mathcal{N}_{pl}\,, \mathcal{N}_{ast} \rangle$$

where $\langle\,,\rangle$ is the Euclidean scalar product.

Let us consider the unit vectors $\mathcal{X}(0), \mathcal{X}'(0)$, pointing to the position of the perihelion of the orbit of the asteroid and the planet respectively; they also are functions of the Keplerian elements:

$$\mathcal{X}(0) = \begin{pmatrix} \cos\Omega\cos\omega - \sin\Omega\sin\omega\cos I \\ \sin\Omega\cos\omega + \cos\Omega\sin\omega\cos I \\ \sin\omega\sin I \end{pmatrix};$$

$$\mathcal{X}'(0) = \begin{pmatrix} \cos\Omega'\cos\omega' - \sin\Omega'\sin\omega'\cos I' \\ \sin\Omega'\cos\omega' + \cos\Omega'\sin\omega'\cos I' \\ \sin\omega'\sin I' \end{pmatrix};$$

and the mutual perihelion arguments ω_M, ω'_M are given by

$$\begin{cases} \cos\omega_M = \langle \mathcal{A}_{nod}\,, \mathcal{X}(0) \rangle \\ \\ \sin\omega_M = \langle \mathcal{A}_{nod} \wedge \mathcal{X}(0)\,, \mathcal{N}_{ast} \rangle \end{cases} \qquad \begin{cases} \cos\omega'_M = \langle \mathcal{A}_{nod}\,, \mathcal{X}'(0) \rangle \\ \\ \sin\omega'_M = \langle \mathcal{A}_{nod} \wedge \mathcal{X}'(0), \mathcal{N}_{pl} \rangle\,. \end{cases}$$

REMARK 21. Let $\mathcal{E}_{\mathcal{K}} = \{a, e, I, \omega, \Omega\}$ and $\mathcal{E}'_{\mathcal{K}} = \{a', e', I', \omega', \Omega'\}$ be the subset of the Keplerian elements of the asteroid and the planet defining the averaged perturbing function. The transformation

$$\{\mathcal{E}_{\mathcal{K}}, \mathcal{E}'_{\mathcal{K}}\} \longrightarrow \{a, e, a', e', \omega_M, \omega'_M, I_M\}$$

from Keplerian to mutual elements is not 1-1; in fact you can rigidly rotate both orbits together around the origin of the fixed reference system and the mutual elements do not change, while there is a change in the Keplerian ones. Notice that equation (9.5) is not defined if the mutual inclination between the two orbits is zero.

We consider the subset $\mathcal{E}_{\mathcal{M}} = \{a, e, \omega_M, \omega'_M, I_M\}$ of the mutual elements depending on the asteroid Keplerian elements. The derivatives of the mutual elements with respect to

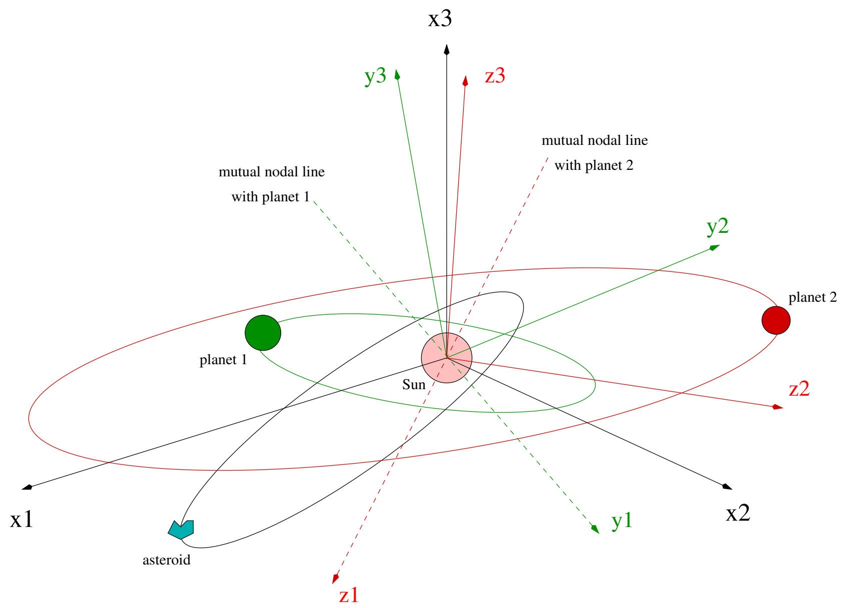

Figure 9.2: Two mutual reference frames y_1, y_2, y_3 and z_1, z_2, z_3 are defined by the pairs of orbits $\{$asteroid, planet 1$\}$ and $\{$asteroid, planet 2$\}$. Also the fixed reference frame x_1, x_2, x_3 is drawn in the figure.

the Delaunay variables appearing in equations (9.4) can be easily computed by the chain rule, passing through the Keplerian elements of the asteroid:

$$\frac{\partial \mathcal{E}_{\mathcal{M}}}{\partial \mathcal{E}_{\mathcal{D}}} = \frac{\partial \mathcal{E}_{\mathcal{M}}}{\partial \mathcal{E}_{\mathcal{K}}} \frac{\partial \mathcal{E}_{\mathcal{K}}}{\partial \mathcal{E}_{\mathcal{D}}} \,;$$

in fact we observe that

$$\frac{\partial \mathcal{E}_{\mathcal{M}}}{\partial \mathcal{E}_{\mathcal{K}}} = \begin{bmatrix} I_2 & O_{2,3} \\ O_{2,3}^t & \dfrac{\partial(\omega_M, \omega'_M, I_M)}{\partial(I, \omega, \Omega)} \end{bmatrix} \quad \text{and} \quad \frac{\partial \mathcal{E}_{\mathcal{K}}}{\partial \mathcal{E}_{\mathcal{D}}} = \begin{bmatrix} \mathcal{M} & O_{2,3}^t \\ O_2 & I_2 \end{bmatrix} \tag{9.6}$$

where I_2 is the 2×2 identity matrix, O_2 and $O_{2,3}$ are respectively the 2×2 and 2×3 zero matrices, and

$$\mathcal{M} = \frac{1}{k\sqrt{a}} \begin{bmatrix} 0 & 0 \\ -\beta/e & 0 \\ \mathrm{cotan} I/\beta & -1/(\beta \sin I) \end{bmatrix}$$

with $\beta = \sqrt{1-e^2}$.

Remark 22. Note that some of the coefficients of the matrix $\mathcal{M}$ becomes singular for $I = 0$. This is a fictitious singularity that can be removed by a rotation of the fixed reference frame.

More generally, given a planetary model with the Sun, N planets ($N \geq 1$) and an asteroid, where the eccentricities and the inclinations of the orbits of the planets are taken into account, we can define N mutual reference frames using the orbits of the asteroid and one planet at a time, provided that each of the mutual inclinations is different from zero.

In Figure 9.2 we show the two mutual reference frames y_1, y_2, y_3 and z_1, z_2, z_3 in the case of four bodies: the Sun, two planets and an asteroid; we also draw the fixed reference frame x_1, x_2, x_3.

When we consider a restricted $N+2$ body problem with $N \geq 1$, we suppose that the motion of the planets is known: this motion can actually be computed, for example, by means of a synthetic theory like in [9]. We can take advantage of the simplified description of the perturbing function R in the mutual reference frames to use a formalism very close to the one of the coplanar case for the planets.

The perturbing function is a sum of terms R_i (with $i = 1 \dots N$) and in each R_i the dependence on the variables of the asteroids and the ones of the only planet i appears. We can study the case of only one perturbing planet: the perturbation of all the planets that we want to include in the model, up to the first order in the perturbing masses μ_i (the ratio between the mass of the planet i and the mass of the Sun), will be obtained as the sum of the contribution of each planet.

From now on we shall consider a system of three bodies: Sun, planet, asteroid. The quantities related to the planet will be distinguished by a prime from the same quantities for the asteroid.

9.5 Geometry of the osculating orbits

In the mutual reference frame, as described in Figure 9.1, we can write the equations of the two osculating orbits of the asteroid and the planet

$$P(u): \begin{cases} x = a[\cos\omega_M(\cos u - e) - \beta\sin\omega_M \sin u] \\ y = a[\sin\omega_M(\cos u - e) + \beta\cos\omega_M \sin u]\cos I_M \\ z = a[\sin\omega_M(\cos u - e) + \beta\cos\omega_M \sin u]\sin I_M \end{cases} \tag{9.7}$$

$$P'(u'): \begin{cases} x' = a'[\cos\omega'_M(\cos u' - e') - \beta'\sin\omega'_M \sin u'] \\ y' = a'[\sin\omega'_M(\cos u' - e') + \beta'\cos\omega'_M \sin u'] \\ z' = 0 \end{cases} \tag{9.8}$$

where $\beta = \sqrt{1-e^2}$, $\beta' = \sqrt{1-e'^2}$ and u, u' are the eccentric anomalies of the asteroid and the planet respectively.

The positive distance $\mathfrak{D}$ between a point on the orbit of the asteroid and a point on the orbit of the planet is defined by its square

$$\begin{aligned} \mathfrak{D}^2(u,u') &= \left|P(u) - P'(u)\right|^2 = a^2(1-e\cos u)^2 + a'^2(1-e'\cos u')^2 - \\ &-2aa'\{[(\cos u - e)\cos\omega_M - \beta\sin u\sin\omega_M][(\cos u' - e')\cos\omega'_M - \beta'\sin u'\sin\omega'_M] + \\ &+\cos I_M[(\cos u - e)\sin\omega_M + \beta\sin u\cos\omega_M][(\cos u' - e')\sin\omega'_M + \beta'\sin u'\cos\omega'_M]\}\ . \end{aligned}$$

For later use we define the function

$$\mathrm{D}(\ell,\ell') = \mathfrak{D}(u(\ell), u'(\ell'))\,.$$

9.5.1 Crossing conditions and values of the anomalies at the ascending node

If the mutual inclination between the two orbits is different from zero, the node crossing conditions between the asteroid and the planet are represented, in terms of the mutual variables, by

$$d^+_{nod} = \frac{a\beta^2}{1+e\cos\omega_M} - \frac{a'\beta'^2}{1+e'\cos\omega'_M} = 0 \qquad (\textit{ascending node crossing})\,;$$

$$d^-_{nod} = \frac{a\beta^2}{1-e\cos\omega_M} - \frac{a'\beta'^2}{1-e'\cos\omega'_M} = 0 \qquad (\textit{descending node crossing})\,.$$

We shall discuss in details the ascending node crossing case, and we shall write down the formulas for the descending node crossing at the end of this chapter, as the theory for this case is formally equivalent to the first one.

The relations

$$a(1-e\cos\overline{u}) = \frac{a\beta^2}{1+e\cos\omega_M}; \quad a'(1-e'\cos\overline{u}') = \frac{a'\beta'^2}{1+e'\cos\omega'_M} \tag{9.9}$$

define the value of the eccentric anomalies $\overline{u},\overline{u}'$ at the ascending node (even if there are no crossings): from (9.9) we obtain

$$\cos\overline{u} = \frac{\cos\omega_M + e}{1+e\cos\omega_M}; \qquad \sin\overline{u} = \frac{-\beta\sin\omega_M}{1+e\cos\omega_M}\,.$$

We also define

$$\overline{\ell} = \overline{u} - e\sin\overline{u}\,,$$

that is the value of the mean anomaly of the asteroid at crossing. These formulas are valid also for the planet by formally replacing $\overline{u}, e, \omega_M, \beta, \overline{\ell}$ with $\overline{u}', e', \omega'_M, \beta', \overline{\ell}'$.

9.6 The Wetherill function

Let us consider the two straight lines r,r' tangent to both ellipses in the ascending nodal points $P(\overline{u}), P'(\overline{u}')$ and parametrized with the mean anomalies ℓ,ℓ' so that $P(u(t))$ and $\mathrm{r}(\ell(t))$ have the same velocities (derivatives with respect to t) in $P(\overline{u})$ and $P'(u'(t))$ and $\mathrm{r}'(\ell'(t))$ have the same velocities in $P'(\overline{u}')$. The equations for r,r' are

$$\mathrm{r}: \begin{cases} x = \overline{x} - \mathcal{F}(\ell-\overline{\ell}) \\ y = \overline{y} + \mathcal{G}\cos I_M(\ell-\overline{\ell}) \\ z = \overline{z} + \mathcal{G}\sin I_M(\ell-\overline{\ell}) \end{cases} \qquad \mathrm{r}': \begin{cases} x' = \overline{x}' - \mathcal{F}'(\ell'-\overline{\ell}') \\ y' = \overline{y}' + \mathcal{G}'(\ell'-\overline{\ell}') \\ z' = 0 \end{cases}$$

where

$$\begin{cases} \mathcal{F} = ae\sin\omega_M\beta^{-1} \\ \mathcal{G} = a(1+e\cos\omega_M)\beta^{-1} \end{cases} \qquad \begin{cases} \mathcal{F}' = a'e'\sin\omega'_M\beta'^{-1} \\ \mathcal{G}' = a'(1+e'\cos\omega'_M)\beta'^{-1} \end{cases}$$

and

$$\overline{x} = \frac{a\beta^2}{(1+e\cos\omega_M)}\,; \quad \overline{y} = \overline{z} = 0\,; \qquad \overline{x}' = \frac{a'\beta'^2}{(1+e'\cos\omega'_M)}\,; \quad \overline{y}' = \overline{z}' = 0\,.$$

REMARK 23. The condition on the velocity at the tangency points can be easily checked using a reparametrization of the straight lines by the parameters $u(\ell), u'(\ell')$, that are related to the mean anomalies ℓ, ℓ' by Kepler's equations

$$\ell = u - e\sin u\,; \qquad \ell' = u' - e'\sin u'\,. \tag{9.10}$$

DEFINITION 20. *We call* Wetherill function *[77] the positive distance* d *between a point on* r *and a point on* r$'$. *The square of this function is*

$$\begin{aligned} \mathrm{d}^2(\ell,\ell') &= (x-x')^2 + (y-y')^2 + z^2 = k^2[\mathcal{F}^2 + \mathcal{G}^2] + k'^2[\mathcal{F}'^2 + \mathcal{G}'^2] - \\ &\quad -2kk'[\mathcal{F}\mathcal{F}' + \mathcal{G}\mathcal{G}'\cos I_M] - 2d^+_{nod}[\mathcal{F}k - \mathcal{F}'k'] + (d^+_{nod})^2 \end{aligned}$$

where $k = k(\ell) = \ell - \overline{\ell}$ *and* $k' = k'(\ell') = \ell' - \overline{\ell}'$.

Note that d^2 is a quadratic form in the variables (k,k'): it is homogeneous when there is a crossing at the ascending node. We can write it in the more concise form

$$\mathrm{d}^2(\ell,\ell') = d^2(\kappa) = \kappa^t \mathcal{A}\kappa + B^t\kappa + (d^+_{nod})^2 \tag{9.11}$$

where

$$\kappa = (k',k)\,; \qquad B = 2(B_1,B_2)\,; \qquad \mathcal{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix};$$

with components

$$\begin{cases} B_1 = d^+_{nod}\mathcal{F}' \\ B_2 = -d^+_{nod}\mathcal{F} \end{cases} \qquad \begin{cases} A_{11} = [\mathcal{F}'^2 + \mathcal{G}'^2] \\ A_{12} = A_{21} = -[\mathcal{F}\mathcal{F}' + \mathcal{G}\mathcal{G}'\cos I_M] \\ A_{22} = [\mathcal{F}^2 + \mathcal{G}^2]\,. \end{cases}$$

For later use we define $\mathfrak{d}^2$ as the squared Wetherill function in coordinates (u,u'):

$$\mathfrak{d}^2(u,u') = \mathrm{d}^2(\ell(u),\ell'(u'))\,.$$

The following proposition holds:

PROPOSITION 9. *The matrix $\mathcal{A}$ is always* positive definite *if $I_M > 0$. If $I_M = 0$ we have degeneracy of $\mathcal{A}$ if and only if the straight lines* r, r$'$ *are parallel: in this case $\mathcal{A}$ is* positive semi-definite.

Proof. $\mathcal{A}$ is a symmetric 2×2 matrix and it is positive definite if and only if its principal invariants, the trace $tr(\mathcal{A})$ and the determinant $det(\mathcal{A})$, are positive. By a direct computation we have

$$\begin{cases} tr(\mathcal{A}) = \frac{a^2}{\beta^2}(1+2e\cos\omega_M+e^2)+\frac{a'^2}{\beta'^2}(1+2e'\cos\omega'_M+e'^2) \\ det(\mathcal{A}) = \mathcal{F}^2\mathcal{G}'^2+\mathcal{F}'^2\mathcal{G}^2+\mathcal{G}^2\mathcal{G}'^2\sin^2 I_M - 2\mathcal{F}\mathcal{F}'\mathcal{G}\mathcal{G}'\cos I = \\ \quad = \frac{a^2a'^2}{\beta^2\beta'^2}\Big\{\Big[e\sin\omega_M\cos I_M(1+e'\cos\omega'_M)-e'\sin\omega'_M(1+e\cos\omega_M)\Big]^2+ \\ \quad +\sin^2 I_M(1+e'\cos\omega'_M)^2(1+2e\cos\omega_M+e^2)\Big\}. \end{cases}$$

From the above expressions we deduce that $tr(\mathcal{A})>0$ and that

$$det(\mathcal{A})=0 \quad \Longleftrightarrow \quad \begin{cases} I_M=0 \\ e\sin\omega_M(1+e'\cos\omega'_M)-e'\sin\omega'_M(1+e\cos\omega_M)=0 \end{cases}$$

that corresponds to the straight lines defined by r, r' being parallel.

□

DEFINITION 21. *We call* tangent crossings *the crossing orbital configurations for which* $det(\mathcal{A})=0$.

In the development of this theory we shall assume that the mutual inclination I_M is different from zero during the whole evolution, or at least in a neighborhood of each crossing between the orbits; this implies that *no tangent crossings occur*.

9.7 Kantorovich's method

We shall use the inverse of the Wetherill function $1/\mathrm{d}$ to extract the principal part from the perturbing function.

The distance D is a 2π-periodic function in both variables ℓ, ℓ' and this property can be used to shift the integration domain $\mathbb{T}^2=\{(\ell,\ell'):-\pi\le\ell\le\pi,-\pi\le\ell'\le\pi\}$ in a suitable way, according to the node where the orbits cross each other, in such a way that the crossing values $(\overline{\ell},\overline{\ell}')$ will be always internal points of this domain.

Using Kantorovich's method we study the regularity properties of the averaged perturbing function $\overline{R}$ defined in (9.3). When the orbits intersect each other, this function is defined by a convergent integral of an unbounded function, in fact the integrand has a first order polar singularity in the points $(\overline{\ell},\overline{\ell}')$, that correspond to collisions.

Let $\mathbb{T}^2=\{(\ell,\ell'):-\pi\le\ell\le\pi,-\pi\le\ell'\le\pi\}$. We shall prove that in computing the derivatives of $\overline{R}$ with respect to the mutual variables, for instance the e-derivatives, we can use the decomposition

$$\frac{(2\pi)^2}{\mu k^2}\frac{\partial}{\partial e}\overline{R} = \int_{\mathbb{T}^2}\frac{\partial}{\partial e}\left[\frac{1}{\mathrm{D}}-\frac{1}{\mathrm{d}}\right]d\ell\, d\ell' + \frac{\partial}{\partial e}\int_{\mathbb{T}^2}\frac{1}{\mathrm{d}}d\ell\, d\ell' \,, \tag{9.12}$$

where μ is the perturbative parameter corresponding to the ratio between the mass of the planet and the mass of the Sun.

9.8 Integration of $1/\mathrm{d}$

Let $\mathbb{T}^2 = \{(\ell,\ell') : -\pi \le \ell \le \pi, -\pi \le \ell' \le \pi\}$. First we move the ascending node crossing point $(\overline{\ell},\overline{\ell}')$ to the origin of the reference system by the translation

$$\tau_{\overline{\ell}',\overline{\ell}}: \ (\ell',\ell) \longrightarrow (k',k) = \kappa \tag{9.13}$$

with $k = \ell - \overline{\ell}, k' = \ell' - \overline{\ell}'$; we call $\overline{\mathbb{T}}^2$ the translated domain. Then we perform a coordinate change to eliminate the linear terms in the squared Wetherill function $d^2(\kappa)$ defined by equation (9.11). We define the inverse of the transformation used for this purpose

$$\Xi^{-1} : \psi \longrightarrow \kappa = \mathcal{T}\,\psi + S$$

where $S = (S_1, S_2) \in \mathbb{R}^2,\ \psi = (y', y) \in \mathbb{R}^2$ are the new variables, and $\mathcal{T}$ is a 2×2 real-valued matrix.

We obtain the vector S by setting to zero the coefficients of the linear terms in the quadratic form in the new variables ψ: the equation to solve is

$$2\,\mathcal{A}S + B = 0 \tag{9.14}$$

whose solutions are

$$\begin{aligned} S_1 &= \frac{B_2A_{12} - B_1A_{22}}{det(\mathcal{A})} = \frac{d^+_{nod}}{det(\mathcal{A})}\mathcal{G}\left[\mathcal{F}\,\mathcal{G}'\cos I_M - \mathcal{F}'\,\mathcal{G}\right] ; \\ S_2 &= \frac{B_1A_{12} - B_2A_{11}}{det(\mathcal{A})} = \frac{d^+_{nod}}{det(\mathcal{A})}\mathcal{G}'\left[\mathcal{F}\,\mathcal{G}' - \mathcal{F}'\,\mathcal{G}\cos I_M\right] . \end{aligned}$$

Furthermore we can determine the non-degenerate matrix $\mathcal{T}$ in order to obtain

$$\mathcal{T}^t\mathcal{A}\mathcal{T} = I_2$$

where I_2 is the 2×2 identity matrix: we set

$$\mathcal{T} = \begin{pmatrix} 1/\tau & -\sigma/\tau\rho \\ 0 & 1/\rho \end{pmatrix}$$

with

$$\tau = \sqrt{A_{11}} = \sqrt{\mathcal{F}'^2 + \mathcal{G}'^2}; \qquad \rho = \sqrt{\frac{det(\mathcal{A})}{A_{11}}} = \sqrt{\frac{det(\mathcal{A})}{\mathcal{F}'^2 + \mathcal{G}'^2}};$$

$$\sigma = A_{12}\sqrt{\frac{1}{A_{11}}} = -\frac{\mathcal{F}\,\mathcal{F}' + \mathcal{G}\,\mathcal{G}'\cos I_M}{\sqrt{\mathcal{F}'^2 + \mathcal{G}'^2}} .$$

The transformation

$$\Xi : \kappa \longrightarrow \psi = \mathcal{R}\left[\kappa - S\right] , \tag{9.15}$$

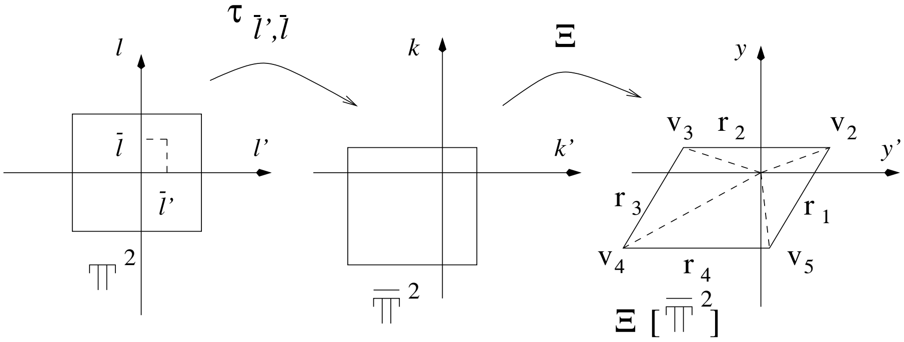

Figure 9.3: Description of the transformations of the domain $\mathbb{T}^2$ after the two coordinate changes that are used to bring the squared Wetherill function into the form $y^2 + y'^2 + (d^+_{min})^2$ in the new variables (y', y). In the parallelogram on the right side we show the decomposition of the domain used for the integration.

where

$$\mathcal{R} = \mathcal{T}^{-1} = \begin{pmatrix} \tau & \sigma \\ 0 & \rho \end{pmatrix}$$

brings the quadratic form $d^2(\kappa)$ into a simpler form in the new variables ψ, and the domain $\overline{\mathbb{T}}^2$ is transformed into a parallelogram with two sides parallel to the y' axis (see Figure 9.3); after performing the change of variables (9.15) we have

$$d^2\left(\Xi^{-1}(\psi)\right) = y^2 + y'^2 + (d^+_{min})^2 \,,$$

where

$$\begin{aligned} d^+_{min} &= |d^+_{nod}|\left\{1 - \frac{1}{det(\mathcal{A})}\left[\mathcal{F}'^2\mathcal{G}^2 - 2\mathcal{F}\mathcal{F}'\mathcal{G}\mathcal{G}'\cos I_M + \mathcal{F}^2\mathcal{G}'^2\right]\right\}^{1/2} = \\ &= |d^+_{nod}|\left[\frac{\mathcal{G}^2\mathcal{G}'^2\sin^2 I_M}{det(\mathcal{A})}\right]^{1/2} \end{aligned} \tag{9.16}$$

is the minimal distance between the two straight lines r and r$'$.

REMARK 24. d^+_{min} is in general smaller than the absolute value of the nodal distance $|d^+_{nod}|$, and it is zero only when $d^+_{nod} = 0$. It can be used as an analytic approximation for the MOID (Minimal Orbital Intersection Distance) [6], [75] when the minimal distance between the two orbits is reached in a neighborhood of the nodal points (usually when the mutual inclination is not small).

Using the variable changes (9.13), (9.15) and the transformation to polar coordinates, whose inverse is

$$\Pi^{-1} : \begin{pmatrix} r \\ \theta \end{pmatrix} \longrightarrow \begin{pmatrix} y' \\ y \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} ,$$

we have

$$\begin{aligned}\int_{\mathbb{T}^2} \frac{1}{\mathrm{d}}\, d\ell\, d\ell' &= \frac{1}{\sqrt{det(\mathcal{A})}} \int_{\Xi\left[\overline{\mathbb{T}}^2\right]} \frac{1}{\sqrt{y^2+y'^2+(d^+_{min})^2}}\, dy\, dy' = \\ &= \frac{1}{\sqrt{det(\mathcal{A})}} \int_{\mathfrak{T}} \frac{r}{\sqrt{r^2+(d^+_{min})^2}}\, dr\, d\theta\end{aligned}$$

where $\Xi^{-1}\left[\Pi^{-1}(\mathfrak{T})\right] = \overline{\mathbb{T}}^2$ and $\overline{\mathbb{T}}^2 = \{(\ell-\overline{\ell}, \ell'-\overline{\ell}') : (\ell,\ell') \in \mathbb{T}^2\}$. Integrating in the r variable we obtain

$$\int_{\mathbb{T}^2} \frac{1}{\mathrm{d}}\, d\ell\, d\ell' = \frac{1}{\sqrt{det(\mathcal{A})}} \cdot \left\{ \sum_{i=1}^{4} \int_{\theta_i}^{\theta_{i+1}} \sqrt{(d^+_{min})^2 + r_i^2(\theta)}\, d\theta - 2\pi d^+_{min} \right\} \tag{9.17}$$

where r_i, with $i = 1 \dots 4$, are the lines delimiting the parallelogram $\Xi[\overline{\mathbb{T}}^2]$, defined by

$$r_1(\theta) = \frac{\rho\tau(\pi-\overline{\ell}'-S_1)}{\rho\cos\theta-\sigma\sin\theta}; \qquad r_2(\theta) = \frac{\rho(\pi-\overline{\ell}-S_2)}{\sin\theta};$$

$$r_3(\theta) = \frac{-\rho\tau(\pi+\overline{\ell}'+S_1)}{\rho\cos\theta-\sigma\sin\theta}; \quad r_4(\theta) = \frac{-\rho(\pi+\overline{\ell}+S_2)}{\sin\theta};$$

$\theta_1 = \theta_5 - 2\pi$ and θ_j, with $j = 2 \dots 5$, are the counter-clockwise angles between the y'-axis and the vertex v_j seen from the origin of the axes (see Figure 9.3). These angles are defined by the following relations:

$$0 < \theta_2 < \theta_3 < \pi < \theta_4 < \theta_5 < 2\pi;$$

$$\tan\theta_2 = \frac{\rho(\pi-\overline{\ell}-S_2)}{\sigma(\pi-\overline{\ell}-S_2)+\tau(\pi-\overline{\ell}'-S_1)}; \quad \tan\theta_3 = \frac{\rho(\pi-\overline{\ell}-S_2)}{\sigma(\pi-\overline{\ell}-S_2)-\tau(\pi+\overline{\ell}'+S_1)};$$

$$\tan\theta_4 = \frac{-\rho(\pi+\overline{\ell}+S_2)}{-\sigma(\pi+\overline{\ell}+S_2)-\tau(\pi+\overline{\ell}'+S_1)}; \quad \tan\theta_5 = \frac{-\rho(\pi+\overline{\ell}+S_2)}{-\sigma(\pi+\overline{\ell}+S_2)+\tau(\pi-\overline{\ell}'-S_1)}.$$

Remark 25. The integrals in formula (9.17) are elliptic and the integrand functions are bounded. These integrals are differentiable functions of the mutual orbital elements $\mathcal{E}_{\mathcal{M}}$, therefore we have only to control the term d^+_{min} in order to study the differentiability of the integral average of $1/\mathrm{d}$.

9.9 Singularities of the $\mathcal{E}_{\mathcal{M}}$-derivatives of $\overline{R}$

We can prove, using methods similar to the ones in Chapter 4, the *boundedness of the remainder function* in a neighborhood of the singular points $\overline{u}, \overline{u}'$ when there is an intersection between the orbits.

We can also prove, using again Kantorovich's method, that the $\{e, \omega_M, \omega'_M\}$-derivatives of the remainder function are always Lebesgue integrable over $\mathbb{T}^2$, with a finite value of the integral, so that the average of the remainder function is differentiable.

Indeed these derivatives can be again computed by exchanging the position of the integral and differential operators as in (9.12). Then we shall see that, if the two orbits intersect each other, there is a discontinuous term in the derivatives of the average of $1/\mathrm{d}$ that is responsible of the discontinuity of the derivatives of $\overline{R}$. These derivatives admit two limit values at crossing (coming from $d^+_{nod} > 0$ and from $d^+_{nod} < 0$).

In particular we can prove the following Theorem, following the same steps of Theorem 14 in Chapter 5:

THEOREM 16. *If there is an ascending node crossing between the orbits at $(u, u') = (\overline{u}, \overline{u}')$, the derivatives of the remainder function $1/\mathfrak{D} - 1/\mathfrak{d}$ with respect to the mutual elements e, ω_M, ω'_M, computed at the node crossing, can be bounded by functions with a first order polar singularity in $\overline{u}, \overline{u}'$, so they are Lebesgue integrable over $\mathbb{T}^2$.*

9.9.1 The $\{e, \omega_M, \omega'_M\}$-derivatives of the average of $1/\mathrm{d}$

As $det(\mathcal{A}) > 0$ and $(\overline{\ell}, \overline{\ell}')$ is in the interior part of $\mathbb{T}^2$, we have $d^2_{min} + r_i^2(\theta) > 0$ for each $\theta \in [\theta_i, \theta_{i+1}]$ and for each $i = 1\ldots4$. Then we can use again the theorem of differentiation under the integral sign and compute, for instance, the derivative of the average of $1/\mathrm{d}$ with respect to e as

$$\frac{\partial}{\partial e}\int_{\mathbb{T}^2}\frac{1}{\mathrm{d}}\,d\ell\,d\ell' = \frac{\partial}{\partial e}\left[\frac{1}{\sqrt{det(\mathcal{A})}}\right]\cdot\left\{\sum_{i=1}^{4}\int_{\theta_i}^{\theta_{i+1}}\sqrt{(d^+_{min})^2 + r_i^2(\theta)}\,d\theta - 2\pi d^+_{min}\right\} +$$
$$+\left[\frac{1}{\sqrt{det(\mathcal{A})}}\right]\cdot\left\{\frac{1}{2}\sum_{i=1}^{4}\int_{\theta_i}^{\theta_{i+1}}\frac{\frac{\partial}{\partial e}[(d^+_{min})^2 + r_i^2(\theta)]}{\sqrt{(d^+_{min})^2 + r_i^2(\theta)}}\,d\theta - 2\pi\frac{\partial}{\partial e}d^+_{min}\right\}\ . \tag{9.18}$$

We have similar formulas for the derivatives with respect to ω_M, ω'_M, obtained simply by substitution of the partial derivative operators (see Appendix C).

The discontinuities present in the terms

$$\frac{\partial}{\partial e}d^+_{min}\,; \qquad \frac{\partial}{\partial \omega_M}d^+_{min}\,; \qquad \frac{\partial}{\partial \omega'_M}d^+_{min}$$

are responsible of the discontinuities in the derivatives of the averaged perturbing function.

9.9.2 Singularities of the I_M-derivative of $\overline{R}$

We observe that we do not need to perform the splitting of Kantorovich's method to compute the derivative with respect to I_M of the integral average of $1/\mathfrak{D}$, in fact we have the following

PROPOSITION 10. *If there is an ascending node crossing, the derivative of* $1/\mathfrak{D}$ *with respect to the mutual inclination* I_M *can be bounded by a function with a first order polar singularity in* $\overline{u},\overline{u}'$*, so it is Lebesgue integrable over* $\mathbb{T}^2$.

Proof. We have

$$\frac{\partial}{\partial I_M}\left[\frac{1}{\mathfrak{D}(\upsilon)}\right] = -\frac{1}{2\mathfrak{D}^3(\upsilon)}\frac{\partial}{\partial I_M}\left[\mathfrak{D}^2(\upsilon)\right],$$

so we need to prove that $\partial\mathfrak{D}^2/\partial I_M$ is an infinitesimal of the second order with respect to $|\nu|$ as $\upsilon \to (\overline{u},\overline{u}')$. For this purpose we only need to check the vanishing of the term

$$2\left\langle \frac{\partial}{\partial I_M}\left[P(\overline{u}) - P'(\overline{u}')\right] - \left[\frac{\partial \overline{u}}{\partial I_M}P_u(\overline{u}) - \frac{\partial \overline{u}'}{\partial I_M}P'_{u'}(\overline{u}')\right], P_u(\overline{u})\, v - P'_{u'}(\overline{u}')\, v' \right\rangle \tag{9.19}$$

that formally represents the first order terms in the derivative of $\mathfrak{D}^2(\upsilon)$ with respect to I_M. The expression in (9.19) vanishes because $P(\overline{u}), P'(\overline{u}'), \overline{u}$ and $\overline{u}'$ do not depend on I_M.

□

REMARK 26. The previous proposition puts in evidence an aspect of the singularities of the derivatives of $\overline{R}$ that also appeared in Chapter 5: there is a variable such that the derivative of $\overline{R}$ with respect to it is a continuous function.

Using a model with all the planets on circular coplanar orbits, the time derivative of the averaged motion of the node does not present loss of continuity at node crossings because the derivative of the perturbing function with respect to the inclination between the asteroid orbits and the common orbital plane of the planets has a first order polar singularity, just like the derivative with respect to I_M in Proposition 10. On the other hand in this case the derivative with respect to the inclination I of the asteroid orbit with respect to a plane of the fixed frame has not necessarily a first order polar singularity because $I = I(I_M, \omega_M, \omega'_M)$, so that the derivatives with respect to both ω_M and ω'_M, that have second order polar singularities, are also involved:

$$\dot{\tilde{\Omega}}(t) = -\frac{1}{k\beta\sqrt{a}\sin I}\left[\frac{\partial \overline{R}}{\partial I}\right] = -\frac{1}{k\beta\sqrt{a}\sin I}\left[\frac{\partial \overline{R}}{\partial I_M}\frac{\partial I_M}{\partial I} + \frac{\partial \overline{R}}{\partial \omega_M}\frac{\partial \omega_M}{\partial I} + \frac{\partial \overline{R}}{\partial \omega'_M}\frac{\partial \omega'_M}{\partial I}\right].$$

We conclude that, if we take into account the eccentricities and the inclinations of the planets, the time evolution of the node $\tilde{\Omega}(t)$ is not necessarily smooth.

9.9.3 Discontinuity in the derivatives of $\overline{R}$

The averaged perturbing function $\overline{R}$ can be regarded as a function of the mutual variables $(a, e, a', e', \omega_M, \omega'_M, I_M)$; the $\{e, \omega_M, \omega'_M\}$-derivatives of $\overline{R}$ are smooth functions in each of the two regions $d^+_{nod} > 0$ and $d^+_{nod} < 0$ and they have a smooth extension onto the boundary of each of these two regions. Correspondingly these derivatives have a twofold definition on the hypersurface $d^+_{nod}(e, e', \omega_M, \omega'_M) = 0$.

The results of the previous sections allow us to write formulas for the differences between the two values of the derivatives of $\overline{R}$ at the ascending node crossing.

DEFINITION 22. *Let*

$$\frac{\partial^+}{\partial e};\ \frac{\partial^+}{\partial \omega_M};\ \frac{\partial^+}{\partial \omega'_M} \qquad \textit{and} \qquad \frac{\partial^-}{\partial e};\ \frac{\partial^-}{\partial \omega_M};\ \frac{\partial^-}{\partial \omega'_M}$$

be the partial derivative operators applied to the restriction to the regions of the space where $d^+_{nod} > 0$ and $d^+_{nod} < 0$ respectively.

We define the operator 'Diff' to describe the differences in the right hand sides of equations (9.2) at $d^+_{nod} = 0$, when we pass from a region where $d^+_{nod} > 0$ to a region where $d^+_{nod} < 0$; this definition is given taking into account the existence of a continuous extension of the derivatives to the boundaries of both sets $\{d^+_{nod} > 0\}$ and $\{d^+_{nod} < 0\}$. We have

$$\begin{aligned}
\mathrm{Diff}\left(\frac{\partial\overline{R}}{\partial G}\right) &= \frac{\partial^-\overline{R}}{\partial G} - \frac{\partial^+\overline{R}}{\partial G} = \\
&= -\frac{\beta}{ke\sqrt{a}}\mathrm{Diff}\left(\frac{\partial\overline{R}}{\partial e}\right) + \frac{\mathrm{cotan}I}{k\beta\sqrt{a}}\left[\frac{\partial\omega_M}{\partial I}\mathrm{Diff}\left(\frac{\partial\overline{R}}{\partial\omega_M}\right) + \frac{\partial\omega'_M}{\partial I}\mathrm{Diff}\left(\frac{\partial\overline{R}}{\partial\omega'_M}\right)\right];\\
\mathrm{Diff}\left(\frac{\partial\overline{R}}{\partial Z}\right) &= \frac{\partial^-\overline{R}}{\partial Z} - \frac{\partial^+\overline{R}}{\partial Z} = -\frac{1}{k\beta\sqrt{a}\sin I}\left[\frac{\partial\omega_M}{\partial I}\mathrm{Diff}\left(\frac{\partial\overline{R}}{\partial\omega_M}\right) + \frac{\partial\omega'_M}{\partial I}\mathrm{Diff}\left(\frac{\partial\overline{R}}{\partial\omega'_M}\right)\right];\\
\mathrm{Diff}\left(\frac{\partial\overline{R}}{\partial g}\right) &= \frac{\partial^-\overline{R}}{\partial g} - \frac{\partial^+\overline{R}}{\partial g} = \left[\frac{\partial\omega_M}{\partial\omega}\mathrm{Diff}\left(\frac{\partial\overline{R}}{\partial\omega_M}\right) + \frac{\partial\omega'_M}{\partial\omega}\mathrm{Diff}\left(\frac{\partial\overline{R}}{\partial\omega'_M}\right)\right];\\
\mathrm{Diff}\left(\frac{\partial\overline{R}}{\partial z}\right) &= \frac{\partial^-\overline{R}}{\partial z} - \frac{\partial^+\overline{R}}{\partial z} = \left[\frac{\partial\omega_M}{\partial\Omega}\mathrm{Diff}\left(\frac{\partial\overline{R}}{\partial\omega_M}\right) + \frac{\partial\omega'_M}{\partial\Omega}\mathrm{Diff}\left(\frac{\partial\overline{R}}{\partial\omega'_M}\right)\right];
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{Diff}\left(\frac{\partial\overline{R}}{\partial e}\right) &= \left[\frac{\partial^-\overline{R}}{\partial e} - \frac{\partial^+\overline{R}}{\partial e}\right] = \Gamma\left(\mathcal{E}_K, \mathcal{E}'_K\right)\mathrm{Diff}\left(\frac{\partial}{\partial e}|d^+_{nod}|\right);\\
\mathrm{Diff}\left(\frac{\partial\overline{R}}{\partial\omega_M}\right) &= \left[\frac{\partial^-\overline{R}}{\partial\omega_M} - \frac{\partial^+\overline{R}}{\partial\omega_M}\right] = \Gamma\left(\mathcal{E}_K, \mathcal{E}'_K\right)\mathrm{Diff}\left(\frac{\partial}{\partial\omega_M}|d^+_{nod}|\right);\\
\mathrm{Diff}\left(\frac{\partial\overline{R}}{\partial\omega'_M}\right) &= \left[\frac{\partial^-\overline{R}}{\partial\omega'_M} - \frac{\partial^+\overline{R}}{\partial\omega'_M}\right] = \Gamma\left(\mathcal{E}_K, \mathcal{E}'_K\right)\mathrm{Diff}\left(\frac{\partial}{\partial\omega'_M}|d^+_{nod}|\right);
\end{aligned}$$

with

$$\Gamma\left(\mathcal{E}_K, \mathcal{E}'_K\right) = \frac{k^2\mu}{(2\pi)^2}\frac{-2\pi}{\sqrt{det(\mathcal{A})}}\left\{1 - \frac{1}{det(\mathcal{A})}\left[\mathcal{F}'^2\mathcal{G}^2 - 2\mathcal{F}\mathcal{F}'\mathcal{G}\mathcal{G}'\cos I_M + \mathcal{F}^2\mathcal{G}'^2\right]\right\}^{1/2}$$

and

$$\mathrm{Diff}\left(\frac{\partial}{\partial e}|d^+_{nod}|\right) = \left[\frac{\partial^-|d^+_{nod}|}{\partial e} - \frac{\partial^+|d^+_{nod}|}{\partial e}\right] = \frac{2a(\cos\omega_M(1+e^2)+2e)}{(1+e\cos\omega_M)^2};$$

$$\mathrm{Diff}\left(\frac{\partial}{\partial \omega_M}|d^+_{nod}|\right) = \left[\frac{\partial^-|d^+_{nod}|}{\partial \omega_M} - \frac{\partial^+|d^+_{nod}|}{\partial \omega_M}\right] = -\frac{2a(1-e^2)e\sin\omega_M}{(1+e\cos\omega_M)^2};$$

$$\mathrm{Diff}\left(\frac{\partial}{\partial \omega'_M}|d^+_{nod}|\right) = \left[\frac{\partial^-|d^+_{nod}|}{\partial \omega'_M} - \frac{\partial^+|d^+_{nod}|}{\partial \omega'_M}\right] = \frac{2a'(1-e'^2)e'\sin\omega'_M}{(1+e'\cos\omega'_M)^2}.$$

9.10 Computation of the weak solutions

We can use the formulas of Section 9.9 to define piecewise smooth solutions of equations (9.2). Let us consider two constant values for the semimajor axes a, a' such that there are no mean motion resonances between the asteroid and the planet, and let us suppose that close approaches are avoided during the evolution of the asteroid. We also assume that the time evolution of the variables e', I', ω', Ω' is known and consider equations (9.2). The derivatives of the averaged perturbing function $\overline{R}$ are continuous functions (indeed smooth) of the mutual variables in each of the connected regions where crossings are avoided; some of these derivatives are subjected to a loss of regularity on the node crossing surfaces.

When an averaged solution $\widetilde{G}(t), \widetilde{Z}(t), \tilde{g}(t), \tilde{z}(t)$ arrives at the ascending node crossing surface $d^+_{nod} = 0$ at time t^*, coming from a region in the phase space such that $d^+_{nod} > 0$, we can take as the new Cauchy problem, defining the future motion, the one obtained by adding the terms described in Section 9.9.3 to the right hand sides of equations (9.2).

Note that if a solution arrives at $d^+_{nod} = 0$ coming from a region where $d^+_{nod} < 0$ we can define the new Cauchy problem by subtracting the same terms.

For very peculiar choices of the initial conditions it is possible that this definition does not give rise to a unique solution, but there are two ways of continuing the solution after the singularity. This happens when the derivatives of d^+_{nod} with respect to the mutual variables vanish at the node crossing.

This theory can be used in a completely analogous way for crossings at the descending node, and it is also suitable to deal with *double crossings*, that is crossings at both ascending and descending node with the same planet.

9.11 The descending node crossing case

Let us assume that a descending node crossing occurs between the orbits, we can follow a procedure similar to the one of the previous case to define weak averaged solution of equations (9.2).

We use the same reference frame as for the ascending crossing case, with the x-axis towards the ascending mutual node in order to be able to use the same formulas also in case of double crossing.

The relations

$$a(1-e\cos\overline{u}_1)=\frac{a\beta^2}{1-e\cos\omega_M};\quad a'(1-e'\cos\overline{u}_1')=\frac{a'\beta'^2}{1-e'\cos\omega_M'} \tag{9.20}$$

define the value of the eccentric anomalies $\overline{u}_1,\overline{u}_1'$ at the descending node (even if there are no crossings): we have

$$\cos\overline{u}_1=\frac{e-\cos\omega_M}{1-e\cos\omega_M};\qquad \sin\overline{u}_1=\frac{\beta\sin\omega_M}{1-e\cos\omega_M}\ .$$

We also define

$$\overline{\ell}_1=\overline{u}_1-e\sin\overline{u}_1\,,$$

that is the value of the mean anomaly of the asteroid at crossing. These formulas are valid also for the planet by formally replacing $\overline{u}_1,e,\omega_M,\beta,\overline{\ell}_1$ with $\overline{u}_1',e',\omega_M',\beta',\overline{\ell}_1'$.

We define the *Wetherill function* using straight lines tangent to the orbits at the descending mutual node. The square of the Wetherill function is given by

$$\begin{aligned}\mathrm{d}_1^2(\ell,\ell') &= (x-x')^2+(y-y')^2+z^2=k_1^2[\mathcal{F}_1^2+\mathcal{G}_1^2]+k_1'^2[\mathcal{F}_1'^2+\mathcal{G}_1'^2]-\\ &\quad -2k_1k_1'[\mathcal{F}_1\mathcal{F}_1'+\mathcal{G}_1\mathcal{G}_1'\cos I_M]+2d_{nod}^-[\mathcal{F}_1k_1-\mathcal{F}_1'k_1']+(d_{nod}^-)^2\end{aligned}$$

where $k_1=\ell_1-\overline{\ell}_1, k_1'=\ell_1'-\overline{\ell}_1'$ and

$$d_{nod}^-=\frac{a(1-e^2)}{1-e\cos\omega_M}-\frac{a'(1-e'^2)}{1-e'\cos\omega_M'}\,;$$

$$\begin{cases}\mathcal{F}_1=ae\sin\omega_M\beta^{-1}\\ \mathcal{G}_1=-a(1-e\cos\omega_M)\beta^{-1}\,;\end{cases}\qquad\qquad \begin{cases}\mathcal{F}_1'=a'e'\sin\omega_M'\beta'^{-1}\\ \mathcal{G}_1'=-a'(1-e'\cos\omega_M')\beta'^{-1}\ .\end{cases}$$

We can write it more concisely as

$$\mathrm{d}_1^2(\ell,\ell')=d_1^2(\kappa_1)=\kappa_1^t\mathcal{A}_d\,\kappa_1+B_d^t\kappa_1+(d_{nod}^-)^2$$

where

$$\kappa_1=(k_1',k_1)\,;\qquad B_d=2(B_1^-,B_2^-)\,;\qquad \mathcal{A}_d=\begin{bmatrix}A_{11}^- & A_{12}^-\\ A_{21}^- & A_{22}^-\end{bmatrix};$$

with components

$$\begin{cases}B_1^-=-d_{nod}^-\mathcal{F}_1'\\ B_2^-=d_{nod}^-\mathcal{F}_1\,;\end{cases}\qquad \begin{cases}A_{11}^-=[\mathcal{F}_1'^2+\mathcal{G}_1'^2]\\ A_{12}^-=A_{21}^-=-[\mathcal{F}_1\mathcal{F}_1'+\mathcal{G}_1\mathcal{G}_1'\cos I_M]\\ A_{22}^-=[\mathcal{F}_1^2+\mathcal{G}_1^2]\ .\end{cases}$$

The variable change to eliminate the linear terms in the quadratic form $d_1^2(\kappa_1)$ is defined by its inverse

$$\Xi_1^{-1}:\psi_1\longrightarrow\kappa_1=\mathcal{T}_1\,\psi_1+S_d$$

where $S_d = (S_1^-, S_2^-) \in \mathbb{R}^2$, $\psi_1 = (y_1', y_1) \in \mathbb{R}^2$ are the new variables and $\mathcal{T}_1$ is a 2×2 real-valued invertible matrix.

The equation to be solved in this case is

$$2\, \mathcal{A}_d S_d + B_d = 0 \tag{9.21}$$

whose solutions are

$$\begin{aligned} S_1^- &= \frac{B_2^- A_{12}^- - B_1^- A_{22}^-}{det(\mathcal{A}_d)} = \frac{d_{nod}^-}{det(\mathcal{A}_d)} \mathcal{G}_1 \left[\mathcal{F}_1' \mathcal{G}_1 - \mathcal{F}_1 \mathcal{G}_1' \cos I_M\right] ; \\ S_2^- &= \frac{B_1^- A_{12}^- - B_2^- A_{11}^-}{det(\mathcal{A}_d)} = \frac{d_{nod}^-}{det(\mathcal{A}_d)} \mathcal{G}_1' \left[\mathcal{F}_1' \mathcal{G}_1 \cos I_M - \mathcal{F}_1 \mathcal{G}_1'\right] . \end{aligned}$$

We obtain

$$\mathcal{T}_1 = \frac{1}{\sqrt{det(\mathcal{A}_d)}} \begin{pmatrix} \rho_1 & -\sigma_1 \\ 0 & \tau_1 \end{pmatrix} = \begin{pmatrix} 1/\tau_1 & -\sigma_1/\tau_1\rho_1 \\ 0 & 1/\rho_1 \end{pmatrix} \tag{9.22}$$

where

$$\tau_1 = \sqrt{A_{11}^-} = \sqrt{\mathcal{F}_1'^2 + \mathcal{G}_1'^2}; \qquad \rho_1 = \sqrt{\frac{det(\mathcal{A}_d)}{A_{11}^-}} = \sqrt{\frac{det(\mathcal{A}_d)}{\mathcal{F}_1'^2 + \mathcal{G}_1'^2}};$$

$$\sigma_1 = A_{12}^- \sqrt{\frac{1}{A_{11}^-}} = -\frac{\mathcal{F}_1 \mathcal{F}_1' + \mathcal{G}_1 \mathcal{G}_1' \cos I_M}{\sqrt{\mathcal{F}_1'^2 + \mathcal{G}_1'^2}};$$

and

$$det(\mathcal{A}_d) = \mathcal{F}_1^2 \mathcal{G}_1'^2 + \mathcal{F}_1'^2 \mathcal{G}_1^2 + \mathcal{G}_1^2 \mathcal{G}_1'^2 \sin^2 I_M - 2\mathcal{F}_1 \mathcal{F}_1' \mathcal{G}_1 \mathcal{G}_1' \cos I .$$

The minimal distance between the straight lines in this case is

$$d_{min}^- = |d_{nod}^-| \left\{ 1 - \frac{1}{det(\mathcal{A}_d)} \left[\mathcal{F}_1'^2 \mathcal{G}_1^2 - 2\mathcal{F}_1 \mathcal{F}_1' \mathcal{G}_1 \mathcal{G}_1' \cos I_M + \mathcal{F}_1^2 \mathcal{G}_1'^2 \right] \right\}^{1/2} .$$

We can define weak averaged solutions of equations (9.2) that go across a descending node crossing by a procedure similar to the one described in the previous sections.

We use the inverse of the Wetherill function $1/\mathrm{d}_1$ to extract the principal part from the direct term of the perturbing function.

In computing the derivatives of $\overline{R}$ with respect to the mutual elements, for instance the e-derivative, we can use the decomposition

$$\frac{(2\pi)^2}{\mu k^2} \frac{\partial}{\partial e} \overline{R} = \int_{\mathbb{T}^2} \frac{\partial}{\partial e} \left[\frac{1}{\mathrm{D}} - \frac{1}{\mathrm{d}_1} \right] d\ell\, d\ell' + \frac{\partial}{\partial e} \int_{\mathbb{T}^2} \left[\frac{1}{\mathrm{d}_1} \right] d\ell\, d\ell'; \tag{9.23}$$

9.11.1 Discontinuity in the derivatives of $\overline{R}$

The averaged perturbing function $\overline{R}$ can be regarded as a function of the mutual variables $(a, e, a', e', \omega_M, \omega'_M, I_M)$; the $\{e, \omega_M, \omega'_M\}$-derivatives of $\overline{R}$ are smooth functions in each of the two regions $d^-_{nod} > 0$ and $d^-_{nod} < 0$ and they have a smooth extension onto the boundary of each of these two regions. Correspondingly these derivatives have a twofold definition on the hypersurface $d^-_{nod}(e, e', \omega_M, \omega'_M) = 0$.

We can write formulas for the differences between the two values of the derivatives of $\overline{R}$ at the descending node crossing (see Appendix C for more details).

Definition 23. *Let*

$$\frac{\partial_1^+}{\partial e};\ \frac{\partial_1^+}{\partial \omega_M};\ \frac{\partial_1^+}{\partial \omega'_M} \qquad \textit{and} \qquad \frac{\partial_1^-}{\partial e};\ \frac{\partial_1^-}{\partial \omega_M};\ \frac{\partial_1^-}{\partial \omega'_M}$$

be the partial derivative operators applied to the restriction to the regions of the space where $d^-_{nod} > 0$ and $d^-_{nod} < 0$ respectively.

We define the operator 'Diff$_1$' to describe the differences in the right hand sides of equations (9.2) at $d^-_{nod} = 0$, when we pass from a region where $d^-_{nod} > 0$ to a region where $d^-_{nod} < 0$. We have

$$\begin{aligned}
\mathrm{Diff}_1\left(\frac{\partial \overline{R}}{\partial G}\right) &= \frac{\partial_1^- \overline{R}}{\partial G} - \frac{\partial_1^+ \overline{R}}{\partial G} = \\
&= -\frac{\beta}{ke\sqrt{a}}\mathrm{Diff}_1\left(\frac{\partial \overline{R}}{\partial e}\right) + \frac{\mathrm{cotan} I}{k\beta\sqrt{a}}\left[\frac{\partial \omega_M}{\partial I}\mathrm{Diff}_1\left(\frac{\partial \overline{R}}{\partial \omega_M}\right) + \frac{\partial \omega'_M}{\partial I}\mathrm{Diff}_1\left(\frac{\partial \overline{R}}{\partial \omega'_M}\right)\right]; \\
\mathrm{Diff}_1\left(\frac{\partial \overline{R}}{\partial Z}\right) &= \frac{\partial_1^- \overline{R}}{\partial Z} - \frac{\partial_1^+ \overline{R}}{\partial Z} = -\frac{1}{k\beta\sqrt{a}\sin I}\left[\frac{\partial \omega_M}{\partial I}\mathrm{Diff}_1\left(\frac{\partial \overline{R}}{\partial \omega_M}\right) + \frac{\partial \omega'_M}{\partial I}\mathrm{Diff}_1\left(\frac{\partial \overline{R}}{\partial \omega'_M}\right)\right]; \\
\mathrm{Diff}_1\left(\frac{\partial \overline{R}}{\partial g}\right) &= \frac{\partial_1^- \overline{R}}{\partial g} - \frac{\partial_1^+ \overline{R}}{\partial g} = \left[\frac{\partial \omega_M}{\partial \omega}\mathrm{Diff}_1\left(\frac{\partial \overline{R}}{\partial \omega_M}\right) + \frac{\partial \omega'_M}{\partial \omega}\mathrm{Diff}_1\left(\frac{\partial \overline{R}}{\partial \omega'_M}\right)\right]; \\
\mathrm{Diff}_1\left(\frac{\partial \overline{R}}{\partial z}\right) &= \frac{\partial_1^- \overline{R}}{\partial z} - \frac{\partial_1^+ \overline{R}}{\partial z} = \left[\frac{\partial \omega_M}{\partial \Omega}\mathrm{Diff}_1\left(\frac{\partial \overline{R}}{\partial \omega_M}\right) + \frac{\partial \omega'_M}{\partial \Omega}\mathrm{Diff}_1\left(\frac{\partial \overline{R}}{\partial \omega'_M}\right)\right];
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{Diff}_1\left(\frac{\partial \overline{R}}{\partial e}\right) &= \left[\frac{\partial_1^- \overline{R}}{\partial e} - \frac{\partial_1^+ \overline{R}}{\partial e}\right] = \Gamma_1\left(\mathcal{E}_K, \mathcal{E}'_K\right)\mathrm{Diff}_1\left(\frac{\partial}{\partial e}|d^+_{nod}|\right); \\
\mathrm{Diff}_1\left(\frac{\partial \overline{R}}{\partial \omega_M}\right) &= \left[\frac{\partial_1^- \overline{R}}{\partial \omega_M} - \frac{\partial_1^+ \overline{R}}{\partial \omega_M}\right] = \Gamma_1\left(\mathcal{E}_K, \mathcal{E}'_K\right)\mathrm{Diff}_1\left(\frac{\partial}{\partial \omega_M}|d^+_{nod}|\right); \\
\mathrm{Diff}_1\left(\frac{\partial \overline{R}}{\partial \omega'_M}\right) &= \left[\frac{\partial_1^- \overline{R}}{\partial \omega'_M} - \frac{\partial_1^+ \overline{R}}{\partial \omega'_M}\right] = \Gamma_1\left(\mathcal{E}_K, \mathcal{E}'_K\right)\mathrm{Diff}_1\left(\frac{\partial}{\partial \omega'_M}|d^+_{nod}|\right);
\end{aligned}$$

with

$$\Gamma_1\left(\mathcal{E}_K, \mathcal{E}'_K\right) = \frac{k^2\mu}{(2\pi)^2}\frac{-2\pi}{\sqrt{det(\mathcal{A}_d)}}\left\{1 - \frac{1}{det(\mathcal{A}_d)}\left[\mathcal{F}_1'^2\mathcal{G}_1^2 - 2\mathcal{F}_1\mathcal{F}_1'\mathcal{G}_1\mathcal{G}_1'\cos I_M + \mathcal{F}_1^2\mathcal{G}_1'^2\right]\right\}^{1/2}$$

and

$$\mathrm{Diff}_1\left(\frac{\partial}{\partial e}|d_{nod}^-|\right) = \left[\frac{\partial_1^-|d_{nod}^-|}{\partial e} - \frac{\partial_1^+|d_{nod}^-|}{\partial e}\right] = -\frac{2a(\cos\omega_M(1+e^2)-2e)}{(1-e\cos\omega_M)^2};$$

$$\mathrm{Diff}_1\left(\frac{\partial}{\partial\omega_M}|d_{nod}^-|\right) = \left[\frac{\partial_1^-|d_{nod}^-|}{\partial\omega_M} - \frac{\partial_1^+|d_{nod}^-|}{\partial\omega_M}\right] = \frac{2a(1-e^2)e\sin\omega_M}{(1-e\cos\omega_M)^2};$$

$$\mathrm{Diff}_1\left(\frac{\partial}{\partial\omega'_M}|d_{nod}^-|\right) = \left[\frac{\partial_1^-|d_{nod}^-|}{\partial\omega'_M} - \frac{\partial_1^+|d_{nod}^-|}{\partial\omega'_M}\right] = -\frac{2a'(1-e'^2)e'\sin\omega'_M}{(1-e'\cos\omega'_M)^2}.$$

9.11.2 Double crossings

In the case of double crossings we use both the inverse of the Wetherill functions $1/\mathrm{d}$ and $1/\mathrm{d}_1$ to extract the principal part from the direct term of the perturbing function.

In computing the derivatives of $\overline{R}$ with respect to the mutual elements, for instance the e-derivative, we can use the decomposition

$$\frac{(2\pi)^2}{\mu k^2}\frac{\partial}{\partial e}\overline{R} = \int_{\mathbb{T}^2}\frac{\partial}{\partial e}\left[\frac{1}{\mathrm{D}} - \frac{1}{\mathrm{d}} - \frac{1}{\mathrm{d}_1}\right] d\ell\, d\ell' + \frac{\partial}{\partial e}\int_{\mathbb{T}^2}\left[\frac{1}{\mathrm{d}}\right] d\ell\, d\ell' + \frac{\partial}{\partial e}\int_{\mathbb{T}^2}\left[\frac{1}{\mathrm{d}_1}\right] d\ell\, d\ell'.$$

9.12 Further development of the theory

We think that it is possible possible to define an extension of the theory where the definition of the Wetherill function is given by choosing straight lines tangent to the local minimum points of the distance between two points on the two orbits, instead of lines tangent to the mutual nodes.

An advantage would be that such a theory should be able to deal with very low inclination cases, for which low values of the distance can be reached quite far from both the mutual nodes.

Obviously such a generalization of the secular theory would result into much more complicated formulas, due to the fact that we are not able to write analytic formulas for the values of the anomalies u, u' in which we want to do the linearization of the orbit.

Chapter 10

Critical points of the distance between two orbits

10.1 Introduction

In the previous chapters we have seen that the osculating orbits play a fundamental role when we study the solutions of the N-body problem with a perturbative theory, and we have also shown how the averaging principle can be used to compute the qualitative evolution of these orbits.

In the case of NEAs the study of the motion of the whole osculating orbits is particularly meaningful for the following reason: even if numerical integrations can be used to compute a solution of the full equations of motion, the chaoticity of the orbits, due to the closeness of the planets, makes the angular position of a NEA on its osculating ellipse the most uncertain parameter when we want to predict its position after some time has elapsed.

Given two celestial bodies, with closed osculating orbits in a heliocentric reference frame, we can ask ourselves if these orbits will cross each other after some time Δt has elapsed, or if the distance between the conics will become smaller than a fixed quantity.

This question is relevant, for instance, to the problem of computing the collision probability between an asteroid and the Earth or between an orbiting space station and the *space debris*, that is fragments of old satellites and other materials orbiting without control.

For this purpose it is very useful to be able to compute the minimum of the distance between these two confocal ellipses, that is generally called the MOID (Minimal Orbital Intersection Distance).

This function is not a regular function of the time: in general it is piecewise regular. A loss of regularity evidently appears when the distance become zero.

We shall study all the *stationary points* (or *critical points*) of the squared distance d^2 between a point on one ellipse and a point on a second ellipse with a focus in common with the first. The MOID corresponds to the points in which the minimum of d^2 is attained. When we consider, as in Figure 10.1, the evolution of two local minimum points of d^2, it

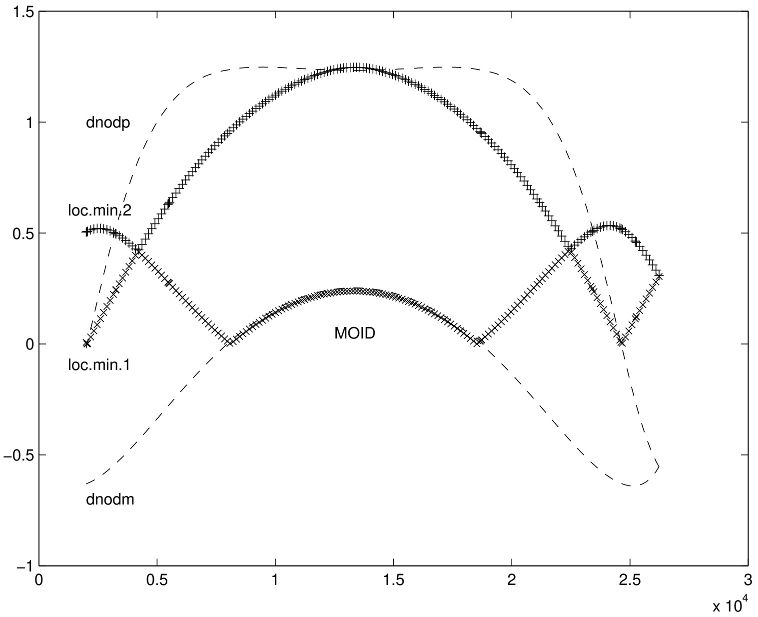

Figure 10.1: The evolution of two local minima (loc.min.1, loc.min.2) and the evolution of the ascending and descending nodal distances are shown in this figure: this example refers to the asteroid 2001 FO_{32}. We can see that the evolutions of the local minima intersect, so that the MOID (the minimum between them) is not a regular function. The loss of regularity appears also when the distance is zero.

can happen that there exists a time $\bar{t}$ for which the two points have the same value of the distance; then these points can exchange their role as lower minimum and these generates a singularity in $\bar{t}$ for the MOID function.

We are also interested in the mathematical problem of giving an upper bound to the total number of the critical points of d^2, and to establish when they can be infinitely many.

We shall show that, if there are finitely many stationary points, then there cannot be more than 16 in the general case and 12 in the case of a circle and an ellipse. We have not proven that this is an *optimal bound* (that is a bound that can be actually reached) and we propose a conjecture for the complete solution of the problem.

We also present an algorithm to compute all the stationary points that is based on the algebraic *theory of elimination* and on the use of the *Fast Fourier Transform*: we have developed a FORTRAN program on the basis of this algorithm and we show some applications of it both to real Keplerian orbits of Near Earth Asteroids and to fictitious cases that result in a high number of stationary points.

The interdisciplinarity of this problem is enhanced: mathematical tools from Celestial Mechanics, Algebra, Numerical Analysis and Algebraic Geometry are used to *attack* the problem from different sides.

10.2 Upper bounds to the number of stationary points

Given two ellipses with a common focus in the space, we define a method to compute all the stationary points of the squared distance function d^2 between a point on the first ellipse and a point on the second one; we use the square of the distance just to avoid *non smoothness* problems that prevent the computation of the partial derivatives in some points of the definition domain.

We try to define the problem with the minimal number of parameters: as it depends only on the mutual position of the two ellipses in space, we use the *mutual reference frame* (x, y, z) defined in Chapter 9.

We recall that the x axis is along the mutual nodal line and that the y axis lies on the plane of the second ellipse, so that this conic is entirely contained in the (x, y) plane (see Figure 9.1 in Chapter 9).

Let a, e, ω_M be the semimajor axis, the eccentricity and the mutual pericenter argument (the counter-clockwise angle between the x axis and the pericenter) of the first ellipse; a', e', ω'_M are the corresponding quantities for the second ellipse and we call I_M the mutual inclination between the two conics.

As the problem is invariant under homotheties, only the ratio a/a' is important, so that we could set, for example, $a' = 1$. However we will continue writing a' for a better understanding of the symmetry in the formulas.

10.2.1 Analytic formulation

As the real parameters u and u' vary in $[0, 2\pi[$, all the points of the two ellipses can be represented in the following way:

$$\begin{cases} x' = a'[\cos\omega'_M(\cos u' - e') - \beta' \sin\omega'_M \sin u'] \\ y' = a'[\sin\omega'_M(\cos u' - e') + \beta' \cos\omega'_M \sin u'] \\ z' = 0 \end{cases} \tag{10.1}$$

$$\begin{cases} x = a[\cos\omega_M(\cos u - e) - \beta \sin\omega_M \sin u] \\ y = a[\sin\omega_M(\cos u - e) + \beta \cos\omega_M \sin u] \cos I_M \\ z = a[\sin\omega_M(\cos u - e) + \beta \cos\omega_M \sin u] \sin I_M\ . \end{cases} \tag{10.2}$$

The squared distance function is given by

$$\begin{aligned} d^2 \;=\;& (x - x')^2 + (y - y')^2 + (z - z')^2 = \\ & A_1 \sin^2 u + A_2 \sin u \cos u + A_3 \cos^2 u + \\ & A_4 \sin^2 u' + A_5 \sin u' \cos u' + A_6 \cos^2 u' + \\ & A_7 \sin u \sin u' + A_8 \sin u \cos u' + A_9 \cos u \sin u' + \\ & A_{10} \cos u \cos u' + A_{11} \sin u + A_{12} \cos u + \\ & A_{13} \sin u' + A_{14} \cos u' + A_{15} \end{aligned}$$

where

$$
\begin{aligned}
&A_1 = a^2\beta^2; \qquad A_2 = 0; \qquad A_3 = a^2;\\
&A_4 = a'^2\beta'^2; \qquad A_5 = 0; \qquad A_6 = a'^2;\\
&A_7 = -2aa'\beta\beta'[\sin\omega_M \sin\omega'_M + \cos I_M \cos\omega_M \cos\omega'_M];\\
&A_8 = -2aa'\beta[-\sin\omega_M \cos\omega'_M + \cos I_M \cos\omega_M \sin\omega'_M];\\
&A_9 = -2aa'\beta'[-\cos\omega_M \sin\omega'_M + \cos I_M \sin\omega_M \cos\omega'_M];\\
&A_{10} = -2aa'[\cos\omega_M \cos\omega'_M + \cos I_M \sin\omega_M \sin\omega'_M];\\
&A_{11} = -2aa'\beta e'[\sin\omega_M \cos\omega'_M - \cos I_M \cos\omega_M \sin\omega'_M];\\
&A_{12} = 2aa'e'[\cos\omega_M \cos\omega'_M + \cos I_M \sin\omega_M \sin\omega'_M] - 2a^2 e;\\
&A_{13} = -2aa'\beta' e[\cos\omega_M \sin\omega'_M - \cos I_M \sin\omega_M \cos\omega'_M];\\
&A_{14} = 2aa'e[\cos\omega_M \cos\omega'_M + \cos I_M \sin\omega_M \sin\omega'_M] - 2a'^2 e';\\
&A_{15} = -2aa'ee'[\cos\omega_M \cos\omega'_M + \cos I_M \sin\omega_M \sin\omega'_M] + a^2e^2 + a'^2e'^2 .
\end{aligned}
$$

The critical points of this function are the real solutions of the system

$$\nabla_{u,u'} d^2(u,u') = 0$$

where $\nabla_{u,u'} d^2$ is the vector of the partial derivatives of d^2 with respect to u, u'. This system can be written, in a more explicit way, as

$$
\begin{cases}
2(A_1 - A_3)\sin u \cos u + A_7 \cos u \sin u' + A_8 \cos u \cos u' \\
-A_9 \sin u \sin u' - A_{10} \sin u \cos u' + A_{11} \cos u - A_{12} \sin u = 0 \\
\\
2(A_4 - A_6)\sin u' \cos u' + A_7 \sin u \cos u' - A_8 \sin u \sin u' \\
+A_9 \cos u \cos u' - A_{10} \cos u \sin u' + A_{13} \cos u' - A_{14} \sin u' = 0
\end{cases}
\tag{10.3}
$$

with $(u,u') \in [0,2\pi[\times[0,2\pi[$. We observe that, due to the periodicity of the trigonometric function of u,u' appearing in the parametrization of the orbits, if $(\overline{u},\overline{u}')$ is a solution of system (10.3) then also $(\overline{u}+2k\pi, \overline{u}'+2h\pi)$, $\forall\,(h,k) \in \mathbb{Z}\times\mathbb{Z}$, are solutions of it.

10.2.2 Algebraic formulation

We can easily find examples of configurations of the two ellipses for which there are an infinite number of critical points: for example the case of two concentric coplanar circles or the case of two overlapping ellipses. We conjecture that if $a, a' > 0$ these are the only two kinds of configurations for which the problem has infinitely many solutions.

Let us suppose that the solutions are only a finite number: we try to give an upper bound for them. We observe that if there are finitely many pairs $(u,u') \in [0,2\pi[\times[0,2\pi[$ solutions of (10.3), then we can find a value $\tilde{u}$ of u and a value $\tilde{u}'$ of u' such that neither $\tilde{u}$, $\tilde{u}+\pi$ nor $\tilde{u}'$, $\tilde{u}'+\pi$ correspond to the u or u' component of a stationary point respectively.

Then we consider the variable change

$$\begin{cases} x = u - \tilde{u} \\ y = u' - \tilde{u}' \end{cases} \tag{10.4}$$

that gives the following relationships between x, y and u, u':

$$\begin{cases} \sin u = \sin x \cos \tilde{u} + \sin \tilde{u} \cos x \\ \cos u = \cos x \cos \tilde{u} - \sin \tilde{u} \sin x \ ; \end{cases} \tag{10.5}$$

$$\begin{cases} \sin u' = \sin y \cos \tilde{u}' + \sin \tilde{u}' \cos y \\ \cos u' = \cos y \cos \tilde{u}' - \sin \tilde{u}' \sin y \ . \end{cases} \tag{10.6}$$

Substituting these equations in (10.3) we obtain two trigonometric polynomials in x, y. Now we use the parametrization

$$\begin{cases} \sin x = \dfrac{2z}{1+z^2} \\ \cos x = \dfrac{1-z^2}{1+z^2} \end{cases} \qquad \begin{cases} \sin y = \dfrac{2w}{1+w^2} \\ \cos y = \dfrac{1-w^2}{1+w^2} \end{cases} \tag{10.7}$$

and we obtain a system of two polynomials in (z, w)

$$\begin{cases} \tilde{p}(z,w) = 0 \\ \tilde{q}(z,w) = 0 \end{cases} \tag{10.8}$$

that have total degree 6; more precisely $\tilde{p}(z,w)$ has degree 4 in z and degree 2 in w, while $\tilde{q}(z,w)$ has degree 2 in z and degree 4 in w.

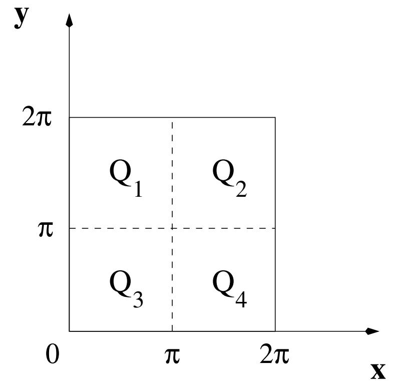

Figure 10.2: The open squares Q_1, Q_2, Q_3, Q_4 represent the sets of validity, in the torus $T = [0, 2\pi[\times[0, 2\pi[$, of the results of Bernstein's theorem on the upper bound to the number of solutions of system (10.3).

Remark 27. The variable change (10.4) has been done in order to be sure that the pairs of angular variables (x, y) that are solutions of system (10.8) lie in the not-connected open set drawn in Figure 10.2, which is composed by four open squares Q_i $(i = 1,..,4)$ defined by

$$Q_1 =]0,\pi[\times]\pi,2\pi[; \qquad Q_2 =]\pi,2\pi[\times]\pi,2\pi[;$$

$$Q_3 =]0,\pi[\times]0,\pi[; \qquad Q_4 =]\pi,2\pi[\times]0,\pi[\;.$$

This is an essential step because the selected parametrization (10.7) makes impossible to search for stationary points corresponding to the lines $\{(\pi,y) : y \in [0,2\pi[\}$ and $\{(x,\pi) : x \in [0,2\pi[\}$ in the torus $[0,2\pi[\times[0,2\pi[$ of all the values that can be assumed by the angular variables x,y; in fact the value π is never assumed either by the x variable or by the y variable as the value $(-1,0)$ is never assumed either by $(\cos x, \sin x)$ or by $(\cos y, \sin y)$: it corresponds to a passage to the limit for $z \to \pm\infty$ or for $w \to \pm\infty$ in formulas (10.7). These excluded values *cannot be* solutions of the system because they correspond to the value $\tilde{u}+\pi$ of the u variable or to the value $\tilde{u}'+\pi$ of the u' variable.

The reason for which we have excluded also the values of (x,y) such that $x = 0$ or $y = 0$ will be clear later.

Bézout's theorem (see [70]) gives us a first bound to the number of solutions: it tells us that if system (10.8) has finitely many common zeros of the form $(x,y) \in \mathbb{C}^2$, then there can be at most 36, that is the product of the total degrees of both polynomials; these solutions are considered with their multiplicity.

We include in this set of solutions also the ones at ∞. A solution at ∞ of system (10.8) is a triple $(z,w,0)$ satisfying the homogenized system obtained from the two polynomials by adding a third variable v.

We can achieve a better bound for the number of solutions by taking into account only the *affine* ones. We state the following:

THEOREM 17. *If there are finitely many stationary points for the squared distance function d^2 (solutions of system (10.3)), then there are at most 16 in the general case and at most 12 if one of the ellipses is a circle.*

Proof. This result will be shown using a theorem due to D. N. Bernstein (see [4], [13], [70]), that we recall here, together with the basic definitions needed to apply this theorem.

DEFINITION 24. *Given a polynomial $r(z,w)$, let n be the number of its terms: we can consider all the pairs (α_i,β_i), $i = 1,..,n$, with integer coefficients, that appear explicitly as multi-index exponents in the terms $z^{\alpha_i}w^{\beta_i}$ of its analytical expression; the convex hull of all these pairs in the plane $\mathbb{R}^2$ is called the Newton polytope of the polynomial $r(z,w)$.*

In Figure 10.3 we draw the Newton polytopes $\mathbf{P},\mathbf{Q}$ for polynomials $\tilde{p}(z,w)$ and $\tilde{q}(z,w)$ that appear in (10.8); we also enhance with circles their corresponding lattices, that describe the exponents of the monomials that explicitly appear in the expression of the two polynomials.

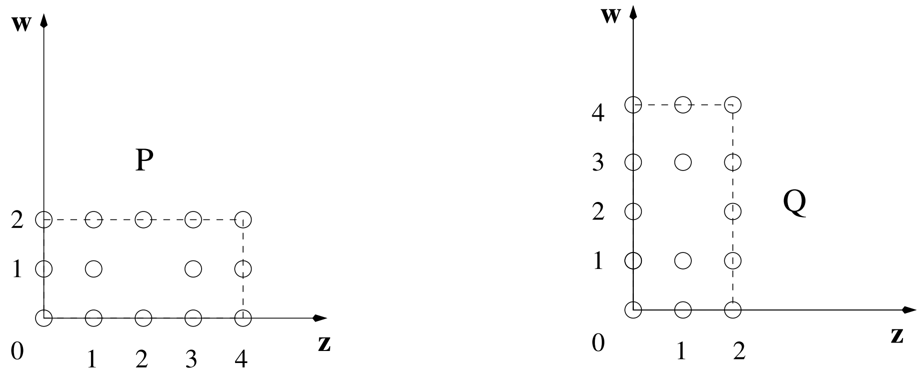

Figure 10.3: Construction of the Newton polytopes $\mathbf{P},\mathbf{Q}$ of the polynomials of system (10.8): the circles with integer coordinates (i,j) in each lattice represent, in the plane (z,w), the terms $z^i w^j$ that explicitly appear in the generic expression of the two polynomials $\tilde{p}(z,w)$ and $\tilde{q}(z,w)$.

DEFINITION 25. *Given two polytopes* $\mathbf{P},\mathbf{Q}$*, we define their Minkowski sum as the convex set:*

$$\mathbf{M} = \mathbf{P} + \mathbf{Q} = \{a + b : a \in \mathbf{P}, b \in \mathbf{Q}\} \ .$$

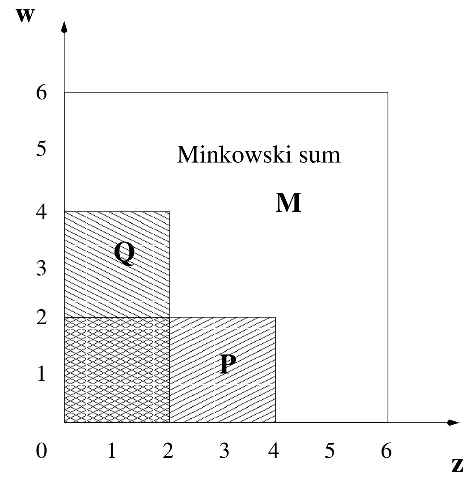

Figure 10.4: The Minkowski sum of the Newton polytopes $\mathbf{P},\mathbf{Q}$: it is generically represented by the square $[0,6]\times[0,6]$ in the plane (z,w).

In Figure 10.4 the *Minkowski sum* of the Newton polytopes $\mathbf{P}$ and $\mathbf{Q}$ is drawn; it is represented by the square $[0,6]\times[0,6]$.

DEFINITION 26. *The mixed area of the Newton polytopes* $\mathbf{P}$ *and* $\mathbf{Q}$ *is defined as*

$$MV_2(\mathbf{P},\mathbf{Q}) = Area(\mathbf{M}) - Area(\mathbf{P}) - Area(\mathbf{Q}) \ .$$

Let $(\mathbb{C}^{\star})^2 = \{(z,w) \in \mathbb{C}\times\mathbb{C} : z,w \neq 0\}$. We can give now the statement of Bernstein's theorem:

THEOREM 18. *(Bernstein) Given two polynomials in two variables, $\tilde{p}(z,w)$ and $\tilde{q}(z,w)$, in the complex field, with finitely many common zeros in $(\mathbb{C}^\star)^2 = \mathbb{C} - \{0\} \times \mathbb{C} - \{0\}$, let* **P** *and* **Q** *be the Newton polytopes in $\mathbb{R}^2$ of $\tilde{p}$ and $\tilde{q}$ respectively. Then the number of common zeros of $\tilde{p}$ and $\tilde{q}$ in $(\mathbb{C}^\star)^2$ is bounded above by the mixed area $MV_2(\mathbf{P},\mathbf{Q})$.*

Bernstein's theorem tells us that the upper bound for the number of solutions of the polynomial system (10.8) in $(\mathbb{C}^\star)^2$ is

$$36 - 8 - 8 = 20\,,$$

and this bound is actually reached for generic values of the parameters, that is for values that do not allow vanishing of the coefficients of $\tilde{p}$ and $\tilde{q}$.

Now we understand the reason to use transformation (10.4) and to be sure that the solutions are far from the lines $\{(0,y) : y \in [0,2\pi[\}$ and $\{(x,0) : x \in [0,2\pi[\}$ in the torus $[0,2\pi[\times[0,2\pi[$: Bernstein's theorem could not take them into account.

The conclusions that we can draw so far are that there are at most 20 solutions (u_j,u'_j) (with $j = 1,..,20$) of the original problem (10.3).

To conclude the proof of the first part of the theorem we have to discard four of the 20 complex solutions: by direct substitution we can check that the pairs

$$(i,i),\ (i,-i),\ (-i,i),\ (-i,-i) \tag{10.9}$$

are always present as common roots of both (10.8); this implies that the number of stationary points, that are *real solutions* of system (10.8), is at most 16.

REMARK 28. The four pure imaginary roots (10.9) have been artificially introduced in the two polynomial systems by the variable change (10.7).

We conclude the proof of the theorem by showing that the upper bound of 16 critical points decreases to 12 if one of the eccentricities e or e' vanishes. Let for example $e = 0$: in this case $\tilde{p}(z,w)$ in (10.8) can be factorized into $(1+z^2)\tilde{r}(z,w)$ for a polynomial $\tilde{r}(z,w)$ that has degree 2 in z and w separately.

Therefore we can apply Bernstein's theorem to the polynomial system (with smaller degree)

$$\begin{cases} (1+z^2)^{-1}\tilde{p}(z,w) = 0 \\ \tilde{q}(z,w) = 0 \end{cases} \tag{10.10}$$

and we obtain 12 as upper bound to the maximal number of the critical points.

The case with $e' = 0$ is similar due to the symmetry of the problem. □

10.2.3 The case of two concentric circles

If both the eccentricities e and e' vanish the problem becomes trivial: the solutions can be analytically computed from system (10.3). If $I_M = 0$ there are infinitely many solutions of the system. If $I_M \neq 0$, due to the symmetry of the configuration, we can set $\omega_M = \omega'_M = 0$, so that system (10.3) reduces to

$$\begin{cases} \sin u \cos u' - \cos I_M \cos u \sin u' = 0 \\ \cos u \sin u' - \cos I_M \sin u \cos u' = 0\,. \end{cases} \tag{10.11}$$

By direct substitution in system (10.3) we can see that the 8 pairs of real numbers (u, u') with $0 \leq u, u' < 2\pi$ such that $(\sin u, \sin u')$ or $(\cos u, \cos u')$ is equal to $(0,0)$ are solutions of (10.11).

There are no additional solutions of this system. This can be proven again with Bernstein's theorem: in this case both polynomials $\tilde{p}$ and $\tilde{q}$ in (10.8) can be trivially factorized, so that the problem is equivalent to computing the real roots of the polynomial system

$$\begin{cases} (1+z^2)^{-1}\tilde{p}(z,w) = 0 \\ (1+w^2)^{-1}\tilde{q}(z,w) = 0\,. \end{cases} \tag{10.12}$$

We conclude this section with some observations: system (10.12) can be written in the simpler form

$$\begin{cases} \tilde{\alpha}(z)w^2 + \tilde{\beta}(z)w - \tilde{\alpha}(z) = 0 \\ \tilde{A}(z)w^2 + \tilde{B}(z)w - \tilde{A}(z) = 0 \end{cases} \tag{10.13}$$

for some second degree polynomials $\tilde{\alpha}, \tilde{\beta}, \tilde{A}, \tilde{B}$ in the z variable; thus the resultant with respect to w is

$$\widetilde{\mathcal{R}es}(z) = -\left[-\tilde{\alpha}(z)\tilde{B}(z) + \tilde{A}(z)\tilde{\beta}(z)\right]^2\,. \tag{10.14}$$

To understand the multiplicity of these solutions we use a geometric interpretation of the stationary points for the squared distance between two curves $\psi(u), \xi(u')$: if $\overline{u}$ and $\overline{u}'$ are the parametric coordinates of two points P, P' that correspond to a stationary point $(\overline{u}, \overline{u}')$ of the squared distance, then the straight line that joins P and P' is orthogonal to both the tangent lines to ψ and ξ in P, P'.

By geometric considerations, in the case of two circles it is obvious that if such a property holds for a pair $(\overline{u}, \overline{u}')$, then it is true also for $(\overline{u}, \overline{u}' + \pi)$. Therefore there are two real values of u' in correspondence of each real component u of a stationary point.

If the circles are in the same plane, for each value of the coordinate u there are two values $u'_1, u'_1 + \pi$ that have this geometric property.

10.2.4 A conjecture about the problem

We have done a large number of numerical experiments, and we have found at most cases with 12 stationary points and only 10 points at most if we set to zero the value of one eccentricity (see Section 10.4). We think that these two numbers actually represent the best upper bounds in the two cases respectively.

In the case of two circles the bound of 8 points is easily reached. We summarize our conjecture about the complete solution of the problem in Table 10.1:

$e \neq 0$	$e' \neq 0$	12 points
$e \neq 0$	$e' = 0$	10 points
$e = 0$	$e' \neq 0$	10 points
$e = 0$	$e' = 0$	8 points

Table 10.1: Conjecture on the maximal number of stationary points (third column) as a function of both the eccentricities e, e' of the two ellipses; in the case with $e = e' = 0$ we can easily compute the 8 points. In all these cases we assume that there is a finite number of solutions of the problem.

10.3 The algorithm set up

Let us suppose that the critical points are finitely many: we want to describe a method to compute all of them. We also suppose that the critical points are such that their u and u' components are different from π. This is done in order to have a simplified formalism: we have written the FORTRAN program that does this computation with this assumption.

Remark 29. The assumption that the u and u' components of the solutions of (10.3) are different from π is necessary if we use parametrization (10.15), because it is impossible to search for such solutions using this parametrization, anyway this restriction is *not* an essential feature of our algorithm; in fact we could first apply the change of variables (10.4), and then use parametrization (10.7), in order to obtain all the solutions.

We can reformulate problem (10.3) into a polynomial form by the variable change

$$\begin{cases} \sin u = \dfrac{2t}{1+t^2} \\ \cos u = \dfrac{1-t^2}{1+t^2} \end{cases} \qquad \begin{cases} \sin u' = \dfrac{2s}{1+s^2} \\ \cos u' = \dfrac{1-s^2}{1+s^2} \end{cases} . \tag{10.15}$$

If we use parametrization (10.15), we have to solve the following bivariate polynomial system:

$$\begin{cases} p(t,s) = \alpha(t)s^2 + \beta(t)s + \gamma(t) = 0 \\ q(t,s) = A(t)s^4 + B(t)s^3 + D(t)s - A(t) = 0 \end{cases} \tag{10.16}$$

in which

$$\begin{aligned}
\alpha(t) &= (A_{11} - A_8) + (4A_1 - 4A_3 + 2A_{10} - 2A_{12})t + \\
&\quad +(-4A_1 + 4A_3 + 2A_{10} - 2A_{12})t^3 + (A_8 - A_{11})t^4 ; \\
\beta(t) &= 2A_7 - 4A_9 t - 4A_9 t^3 - 2A_7 t^4 ; \\
\gamma(t) &= (A_{11} + A_8) + (4A_1 - 4A_3 - 2A_{10} - 2A_{12})t + \\
&\quad +(-4A_1 + 4A_3 - 2A_{10} - 2A_{12})t^3 - (A_8 + A_{11})t^4 ; \\
A(t) &= -(A_9 + A_{13}) - 2A_7 t + (A_9 - A_{13})t^2 ; \\
B(t) &= (-4A_4 + 4A_6 - 2A_{10} - 2A_{14}) - 4A_8 t + \\
&\quad +(-4A_4 + 4A_6 + 2A_{10} - 2A_{14})t^2 ; \\
D(t) &= (4A_4 - 4A_6 - 2A_{10} - 2A_{14}) - 4A_8 t + \\
&\quad +(4A_4 - 4A_6 + 2A_{10} - 2A_{14})t^2 .
\end{aligned}$$

We shall use the *resultant theory*, a branch of the algebraic *elimination theory*, to reduce the problem to the computation of the roots of a univariate polynomial, called the resultant.

Let us select a variable, say s; we know that the polynomials $p(t,s), q(t,s)$ in system (10.16) have a common factor, whose degree in s is greater or equal to one, if and only if the value of t is such that the resultant $Res(t)$ of $p(t,s)$ and $q(t,s)$ with respect to s, which is a univariate polynomial in t, is equal to zero.

DEFINITION 27. *Given the polynomial system (10.16), the resultant $Res(t)$ of p and q with respect to the s variable is the polynomial defined as the determinant of the Sylvester matrix*

$$S_0 = \begin{pmatrix} \alpha & 0 & 0 & 0 & A & 0 \\ \beta & \alpha & 0 & 0 & B & A \\ \gamma & \beta & \alpha & 0 & 0 & B \\ 0 & \gamma & \beta & \alpha & D & 0 \\ 0 & 0 & \gamma & \beta & -A & D \\ 0 & 0 & 0 & \gamma & 0 & -A \end{pmatrix} \tag{10.17}$$

where $\alpha, \beta, \gamma, A, B, D$ are the polynomials in the t variable described before.

It comes out from explicit computation that in general $Res(t)$ is a 20th degree polynomial, whose analytical expression is

$$\begin{aligned} Res(t) \;=\;& \alpha^4A^2 + 4\,\alpha\gamma\beta^2A^2 + \beta\,\alpha^3DA + \beta^3\alpha BA - 2\,\alpha^2D\gamma^2B - 2\,\alpha^2A^2\gamma^2 + \\ & + A^2\gamma^4 - \beta^4A^2 + \gamma^3\alpha B^2 + \gamma\alpha^3D^2 - 3\,\alpha^2A\gamma\beta B - A\gamma^3\beta B - \\ & - \beta^3D\gamma A + 3\,\beta\,\alpha DA\gamma^2 + \beta^2\alpha D\gamma B\ . \end{aligned}$$

The real roots of $Res(t)$ represent all the values of t that may correspond to real solutions of (10.16) and we can use this algebraic approach to solve the problem of finding all the stationary points.

REMARK 30. Note that we could collect the terms of the two polynomials in a different way, and obtain a system of the form

$$\begin{cases} p(t,s) = A^*(s)t^4 + B^*(s)t^3 + D^*(s)t - A^*(s) = 0 \\ q(t,s) = \alpha^*(s)t^2 + \beta^*(s)t + \gamma^*(s) = 0 \end{cases}$$

where A^*, B^*, D^* are second degree polynomials in s while $\alpha^*, \beta^*, \gamma^*$ have degree four.

Computing the resultant with respect to t we obtain a polynomial $Res^*(s)$ that has at most 20th degree in s.

We have to notice that the expression of the resultant, after computing the determinant of S_0 in (10.17) and after substituting all the dependencies on the ellipses parameters, is huge. In our algorithm we avoid writing down its complete parameter–dependent expression; we compute the resultant inside a computer program after the assignment of the values to the parameters.

Interpolation methods are standard tools in Numerical Analysis and some interpolation techniques and applications to the problem of solving roots of polynomials can be found in [84], [44].

This algorithm computes the coefficients of the resultant polynomial by an *evaluation - interpolation* method based on the DFT (*Discrete Fourier Transform*) and the IDFT (*Inverse Discrete Fourier Transform*) respectively (see [67]).

REMARK 31. Using the symbolic computation program *MAPLE V* (see [10]) and the basic properties of determinants, we have also checked that in the general case the 20th degree polynomial $\widetilde{\mathcal{R}es}(z)$, resultant of $\tilde{p}(z,w)$ and $\tilde{q}(z,w)$ with respect to w, can always be divided by $(1+z^2)^2$ (this obviously implies that $Res(t)$ can be divided by $(1+t^2)^2$). Therefore the problem reduces to computing the roots of a 16th degree polynomial $\widetilde{\mathcal{R}es}(z)$.

It is worth to mention that recently Kholshevnikov and Vassiliev (see [37]) have used Groebner bases theory to get an 8th degree trigonometric univariate polynomial whose real roots (that are at most 16) play the same role as the ones of $\widetilde{\mathcal{R}es}(z)$.

The result on the upper bound can also follow from both these formulations of the problem: in the generic case (that is for almost all values of the coefficients) each solution of these univariate polynomials corresponds to a unique solution of (10.3) and if the number of (complex) solutions exceeds the generic number of them, then the solutions are infinitely many.

Anyway we would like to emphasize that the proof given in this section is based on Bernstein's theorem only. The symmetry properties of the problem are also enhanced by the polytope construction: indeed the polynomials in (10.8) are not symmetric, but their Newton polytopes are.

10.3.1 Description of the algorithm

In this section we explain the steps of the algorithm:

1. Evaluation of the 36 polynomials that are the coefficients of the matrix S_0 at all the 32th roots of unit
$$\omega_{Mk} = e^{-2\pi i \frac{k}{32}} \qquad k = 0,..,31$$
by a DFT algorithm. Indeed the number of polynomials to evaluate is only 6, because this is the number of the different coefficients in S_0. We need 32 evaluations for each of these 6 polynomials because we know *a priori* that the degree of the resultant, whose coefficients we want to compute, is at most 20, and, applying the DFT, we need to work with a number of evaluations that is a power of 2 and is greater than the degree of the resultant plus 1 (the number of the polynomial coefficients to be determined).

2. Computation of the determinant of the 32 Sylvester matrices; each of them is evaluated at a different point of the complex plane. If a square matrix has its coefficients depending on a variable t, then the evaluation at a point $\bar{t}$ of the determinant of this

matrix is equal to the determinant of the matrix whose coefficients are evaluated at $\bar{t}$; so we have obtained the values of the resultant polynomial in the 32th roots of unit.

3. Application of an IDFT to obtain the coefficients of the resultant from its 32 evaluations.

4. Computation of the real roots of the resultant polynomial. For this point we have used the algorithm described in [5], based on simultaneous iterations.

5. Given a solution $\bar{t} \in \mathbb{R}$ of $Res(t) = 0$, search for one or more values $\bar{s} \in \mathbb{R}$ for which $(\bar{t}, \bar{s})$ is a solution of (10.16). This is perhaps the most *delicate* step of the algorithm: in most cases for each root $\bar{t}$ of $Res(t)$ there is only one root $\bar{s}$ of $Res^*(s)$ such that the pair $(\bar{t}, \bar{s})$ satisfies (10.16), but sometimes this is not true (see Section 10.5.2).

6. Detection of the type of singularity that has been found: the classification in minimum, maximum or saddle points is done by computation of the Hessian matrix $H(u, u')$ of d^2 and of its *determinant* and *trace* at the critical points.

10.4 Examples of applications of the algorithm

We present some applications of the algorithm described in Section (10.3.1): we have written a FORTRAN program for the computation of all the critical points of $d^2(u, u')$ following the steps of the previous section. The figures have been produced using the software package MATLAB.

10.4.1 Examples with high number of stationary points

In this section we show two examples with the maximal number of stationary points that we have found with numerical experiments; both these examples are essential to show a lower bound to the maximal number of points that we have to expect in the general case and when one of the eccentricities vanishes.

The example in Figure 10.5 shows the level lines of the squared distance function d^2 in the plane (u, u') (the angular values are in degrees): one of the ellipses is indeed a circle ($e' = 0$) and there are 10 real solutions of (10.16). In this figure the position of the critical points is enhanced with crosses (local extrema) and asterisks (saddle points). This example refers to the data of Table 10.2 in the mutual reference frame.

In Table 10.3 we describe all the stationary points related to this example. This table contains their angular values u and u' in degrees, the corresponding values of the distance d and the classification as maximum, minimum and saddle points.

The second example (see Figure 10.6) is one of the cases with the highest number of stationary points that we have found: there are 12 real solutions of (10.16). The parameters used in this case, in the mutual reference frame, are described in Table 10.4, while in Table 10.5 we describe all the points related to this example.

a	a'	e	e'	ω_M	ω'_M	I_M
1.2	1.0	0.6	0.0	176.10226°	344.12617°	60.00581°

Table 10.2: Ellipse parameters that correspond to the example of 10 stationary points; one of the two conics is a circle.

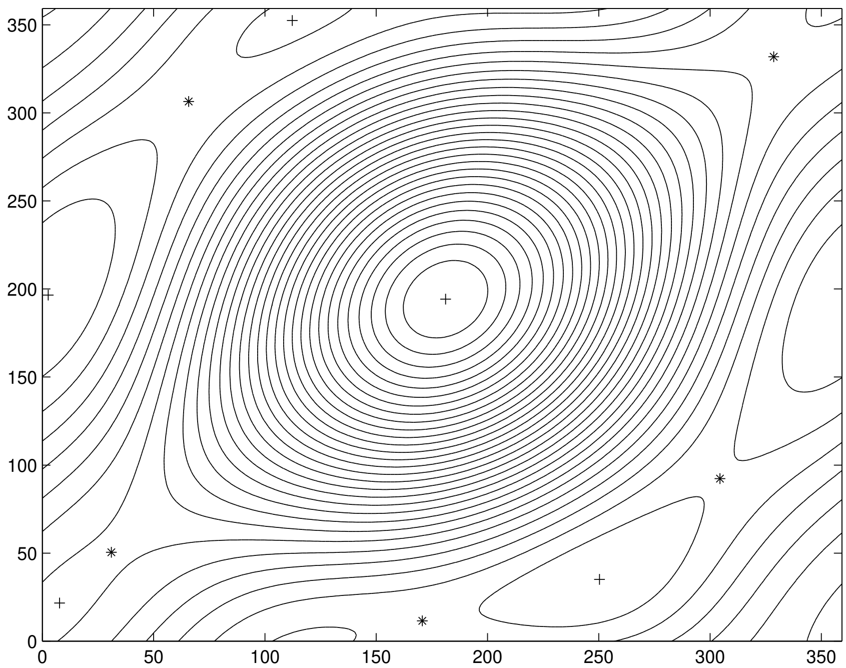

Figure 10.5: Example with 10 stationary points. The level lines of the squared distance function are plotted in the (u, u') plane (both angles are in degrees): there are two maxima, three minima and five saddle points.

u	**u'**	**Distance**	**Type**
2.63347	196.55703	0.51944	MINIMUM
250.32274	35.10467	0.75828	MINIMUM
112.30766	352.40721	0.86325	MINIMUM
304.45454	92.32415	0.90496	SADDLE
170.71827	11.57758	0.92786	SADDLE
65.74317	306.47437	0.93188	SADDLE
328.56116	331.85177	1.44620	SADDLE
30.97048	50.48656	1.47311	SADDLE
7.69205	21.63891	1.48162	MAXIMUM
181.13937	194.20871	2.91903	MAXIMUM

Table 10.3: Description of all the angular components (in degrees) of the example with 10 stationary points, together with the corresponding value of the distance d and the type of singularity.

a	a'	e	e'	ω_M	ω'_M	I_M
1.2	1.0	0.615	0.415	176.17119°	8.15365°	80.00146°

Table 10.4: Parameters for the example of 12 stationary points, the maximal number detected with these numerical experiments.

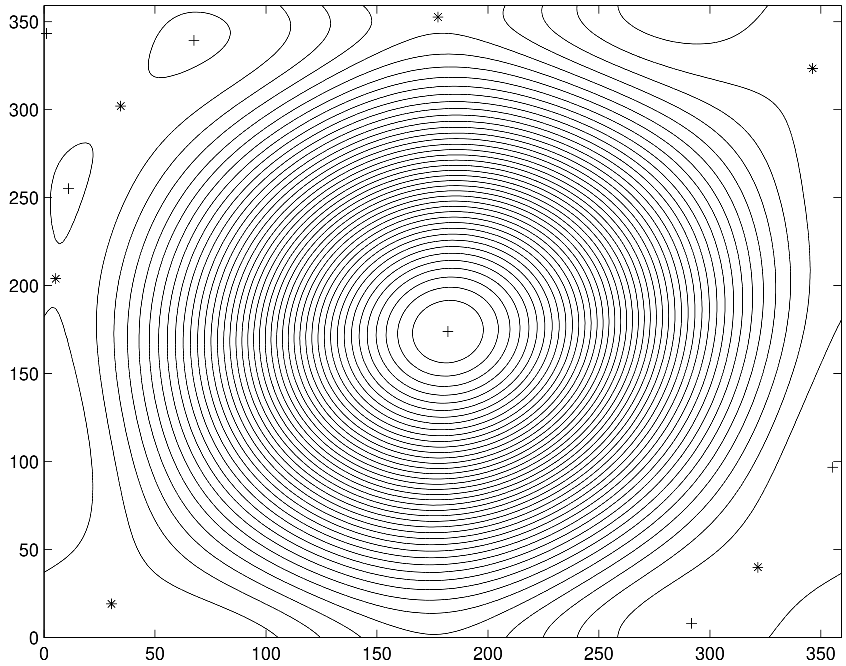

Figure 10.6: Example with 12 stationary points: there are two maxima, four minima and six saddle points.

u	u'	Distance	Type
355.36450	96.89415	0.83241	MINIMUM
291.75411	8.23072	0.86963	MINIMUM
321.61082	40.03329	0.89822	SADDLE
67.55157	339.55278	0.94556	MINIMUM
11.06387	255.14431	0.95536	MINIMUM
5.28161	204.02257	0.97033	SADDLE
34.55134	302.15842	0.97573	SADDLE
30.39864	19.20128	1.03079	SADDLE
346.32545	323.53696	1.04872	SADDLE
1.13537	343.45933	1.05266	MAXIMUM
177.49506	352.69373	1.35279	SADDLE
181.95289	173.95008	3.34644	MAXIMUM

Table 10.5: Description of the angular components of the example with 12 stationary points, together with the corresponding value of the distance d and the type of singularity.

10.4.2 The critical points for the distance Earth–asteroid

We can find a straightforward application of this algorithm in the field of Celestial Mechanics; in fact the ellipses can be thought as osculating orbits of two objects around the Sun (the common focus), that is, each of them is the solution of a two body problem in which the considered bodies are one of the objects and the Sun. In this case the algorithm is useful to study the possibility of collision with the Earth.

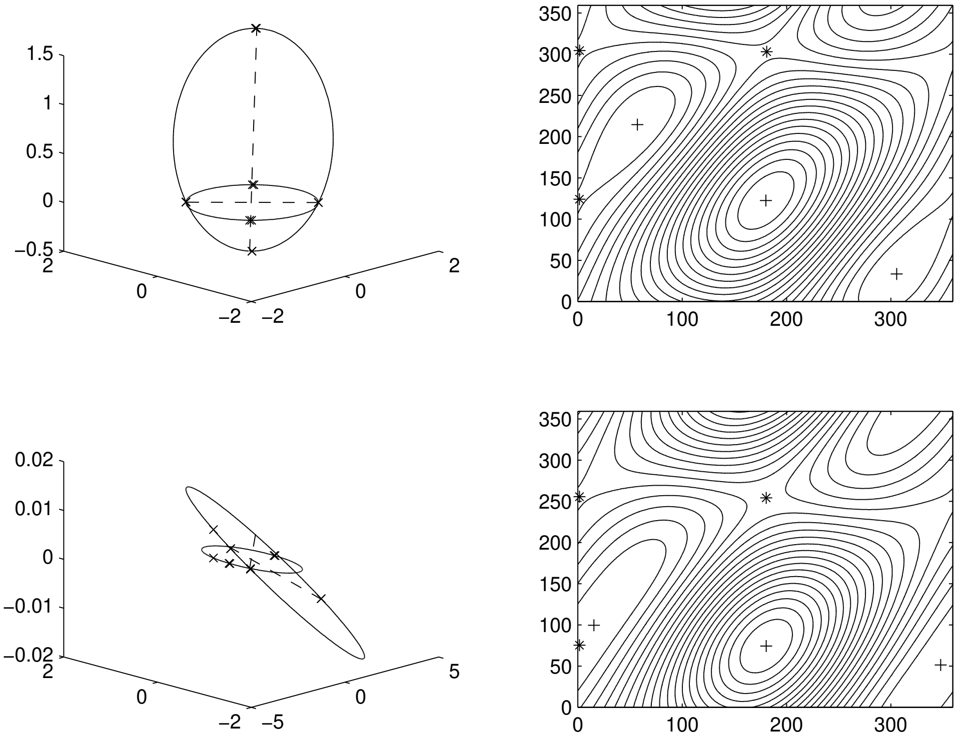

Figure 10.7: Stationary points for the orbits of two Near Earth Asteroids. The first figure (top left) represents the orbit of 1999 AN_{10} and the orbit of the Earth: the values corresponding to the critical points are marked on each orbit. The two dashed lines represent the mutual nodal line and the apsidal line of the asteroid. The second figure (top right) represents the level lines of the squared distance between this asteroid and the Earth in the plane (u,u'), just as in Figure 10.5. The two figures at the bottom are referred to 1994 GV; note that the scale on the vertical axis has changed with respect to the previous asteroid, in fact 1994 GV has an inclination of about 0.457 degrees only.

Using the database of the web site NEODyS (the Near Earth Objects Dynamic Site)

`http://newton.dm.unipi.it/neodys`

we compute all the stationary points of the squared distance function for the orbits of all the known Near Earth Asteroids (more than 1800) and the orbit of the Earth; the database of this site is updated every day and the program computing the stationary points provides

u	u'	Distance	Type
57.22436	214.88127	0.00009	MINIMUM
305.40415	33.59185	0.00416	MINIMUM
1.60052	124.38821	0.66263	SADDLE
1.69017	304.64556	1.53678	SADDLE
180.95797	303.22435	1.64606	SADDLE
180.21098	122.72000	3.12126	MAXIMUM

a	1.45866
a'	1.00089
e	0.56215
e'	0.01697
ω	268.29177°
ω'	145.67486°
I	39.93216°

Table 10.6: Tables of the stationary points for the asteroid 1999 AN_{10}, together with the ellipse parameters used for the computation.

u	u'	Distance	Type
15.86170	100.69515	0.00025	MINIMUM
347.30674	50.83805	0.00589	MINIMUM
1.54197	75.66297	0.03179	SADDLE
1.62360	255.79893	1.96987	SADDLE
180.51300	254.50274	2.02637	SADDLE
180.51009	74.49829	4.02787	MAXIMUM

a	1.99821
a'	1.00089
e	0.51723
e'	0.01697
ω	154.07257°
ω'	80.16853°
I	0.45686°

Table 10.7: Tables of the stationary points for the asteroid 1994 GV, together with the ellipse parameters used for the computation.

a daily computation of these points for all the new orbits and for the ones whose data have changed thanks to additional observations.

From the following two examples it also comes out that not only the absolute minimum of the distance has to be taken into account to estimate the collision probability of a NEA, but all the local minima and, in some cases, also saddle points provide useful information.

In Figure 10.7 we can see the geometry of the orbit of the asteroids 1999 AN_{10} (top left) and 1994 GV (bottom left) with respect to the orbit of the Earth. Also the level lines of the squared distance are plotted (top right and bottom right respectively).

The asteroid 1999 AN_{10} has been discovered in January 1999 and has been intensively studied as a possible Earth impactor for the years 2039 (see [51]) and 2044 (see [52]); it reaches its local minima of the distance in a neighborhood of both the nodal points, the ascending and the descending one, and these values are relatively small, as can be seen from Table 10.6 in which the distance is given in Astronomical Units (1 AU $\simeq$ 150,000,000 Km).

Now we show a similar table for the asteroid 1994 GV. Another aspect of the possible orbital configurations is put in evidence: not only minimum but also saddle points have to be taken into account to understand if an asteroid can be dangerous; in fact the value of the distance displayed in Table 10.7 for the first saddle point (0.03179 AU) is quite small, so that there is an arc of dangerous points joining both minima and passing through this saddle point in the plane (u, u'). This object has a very low inclination (about 0.45 degrees) with respect to the ecliptic plane, so that the minima are not necessarily in a neighborhood of the nodal points like in the previous example.

10.5 Difficult cases and possible improvements of the algorithm

In this section we present the most relevant problems that appear using the algorithm defined in Section 10.3 and we give some suggestions about the improvements that can be done to it.

10.5.1 The degeneration of the polynomial $Res(t)$

There are some values of the ellipse parameters that result in a degeneration of the polynomial $Res(t)$, in the sense that its leading coefficient vanishes, so that its degree decreases. We give a list of the degenerating cases that we have detected using MAPLE V to compute and factorize the resultant:

- if $I_M = 0$ and $(\omega_M - \omega'_M) = k\pi$ $(k \in \mathbb{Z})$, the degree of $Res(t)$ is 18 and the resultant has a factor $t^2(1+t^2)^2$;
- if $I_M = \pi/2$ and $\omega_M = k\pi$, the degree of $Res(t)$ is 16 and the resultant has a factor $t^4(1+t^2)^2$;
- if $I_M = \pi/2$ and $\omega'_M = k\pi$, the degree of $Res(t)$ is 16 and the resultant has a factor $(1+t^2)^2$;
- if $e = 1$, the degree of $Res(t)$ is 16 and the resultant has a factor $t^4(1+t^2)^2$.

The decrease of degree of the polynomial corresponds to the fact that some of its roots go to infinity: this can happen also due to the particular parametrization (10.15) selected, as we already explained in Section 10.2.2.

10.5.2 Multiple correspondence to the roots of $Res(t)$

Once the solutions of $Res(t) = 0$ have been computed and we have to compute the corresponding values of the s variable, then we have to face an additional problem: there can be more than one point $\bar{s}$ corresponding to a given value of $\bar{t}$ such that the pair $(\bar{t}, \bar{s})$ satisfies equations (10.16).

Indeed, given a point $\bar{t}$, the polynomials $p(\bar{t}, s)$ and $q(\bar{t}, s)$ have degree two and four respectively in s, so that we can have two solutions in s, corresponding to the same $\bar{t}$, if the relationship

$$p(\bar{t}, s) \mid q(\bar{t}, s)$$

(q is divisible by p) holds.

In Figure 10.8 a frame sequence of the level lines of the squared distance d^2 is shown in the plane (u, u'), in which two saddle points $(u_1, u'_1), (u_2, u'_2)$, described by two crosses, move in the plane with slow changes of the ellipse parameters until the corresponding value of the difference variable $(u_1 - u_2)$ changes its sign. As the stationary points move

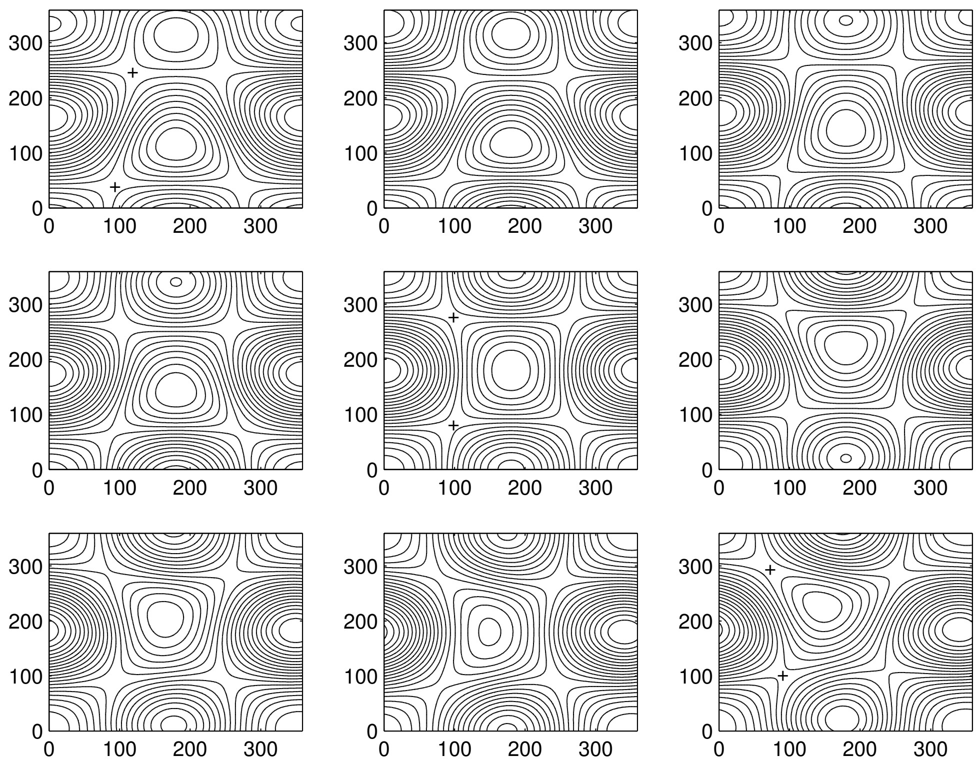

Figure 10.8: In this figure there are nine frames showing a possible evolution of the level lines of d^2 in the plane (u,u'); passing from the first frame to the last, the two saddle points, enhanced with crosses, have the same value of one coordinate in the plane (u,u') (see frame 5).

a	a'	e	e'	ω_M	ω'_M	I_M
1.0	1.0	0.8	0.2	0°	180°	90°

Table 10.8: Parameters for the example in Figure 10.9.

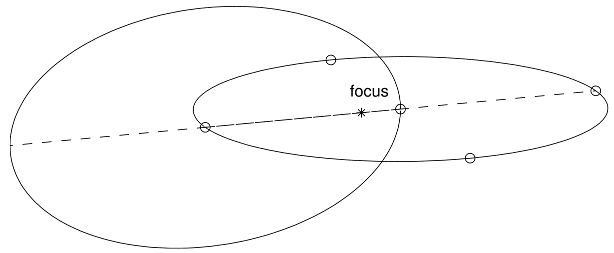

Figure 10.9: A particular configuration between the ellipses: the symmetry center of the first ellipse corresponds to the pericenter of the second one and they are mutually orthogonal. The four critical values of one angular variable that correspond to the same critical value of the other angle are enhanced with circles in this figure, where the asterisk represents the common focus of both the conics.

continuously in this plane, there are values of the parameters for which these points have the same value of the u component and two different values of u'.

A further difficulty is given by the possible vanishing of $p(\bar{t}, s)$; in this way we can have up to four solutions in s, corresponding to $\bar{t}$, as you can see from the example in Figure 10.9 whose corresponding parameters are written in Table 10.8.

In this example the two ellipses are mutually orthogonal and the eccentricities have been selected in a way that the pericenter of a conic coincides with the symmetry center of the other one.

A powerful algebraic tool to deal with this kind of difficulties could be *subresultants theory* (see Appendix D).

The multiple correspondence problem is strictly connected to the multiplicity of the roots of $Res(t)$, as we have already seen for a particular case in Section 10.2.3. While dealing with numerical methods to find the roots of a polynomial, a difficulty is just to detect when there are multiple roots: subresultant theory could give us a tool for their detection, so that we can know *a priori* when we have to expect multiple roots and then modify Step 5 of the algorithm.

Chapter 11

Regularization of the 2-body problem with a smoothed potential

11.1 Introduction

We study the regularization of the collisions in a 2-body problem where the two particles, with positions $q_1, q_2 \in \mathbb{R}^3$, interact with a potential proportional to $1/|q_1 - q_2|^\alpha$, $\alpha > 0$, that is we investigate the possibility to extend the solutions beyond the singularities. Reducing the problem to the plane motion of a point particle in a central force field with potential

$$V_\alpha(x) := \frac{1}{|x|^\alpha}, \qquad x \in \mathbb{R}^2 \setminus \{0\},$$

we start from a solution $u \in C^2([0,T[;\mathbb{R}^2)$ of

$$\begin{cases} \ddot{u} = -\dfrac{\alpha u}{|u|^{(\alpha+2)}}, \qquad t \in]0,T[, \\ \\ (u(0),\dot{u}(0)) = (q_0, p_0) \in (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^2, \end{cases} \tag{11.1}$$

where T is the maximal time of existence of the solution. Note that for $\alpha = 1$ problem (11.1) reduces to the Kepler problem. We are interested in finding global extensions of the solutions of (11.1), that go beyond the collision singularities.

It is well known (see [45]) that the only singularities of the solutions of (11.1) are collisions between the particles, i.e. if $0 < T < +\infty$, then necessarily $u(t) \to 0$ as $t \uparrow T$. Moreover, assuming that the total energy integral $H_{\alpha,0}$ of the system is negative and denoting by I_α the set of all the initial data in (11.1) leading to a collision, we have the following alternatives: if $0 < \alpha < 2$ and $(q_0, p_0) \in I_\alpha$, then $q_0 \wedge p_0 = 0$ (that is the angular momentum is necessarily zero); if $\alpha \geq 2$ then all the initial conditions (q_0, p_0) lead to a collision, as the orbital energy is negative (see Chapter 2).

Motivated by the discussion in [14], in order to regularize the collision solutions of (11.1) our first step consists in replacing the potential V_α with the smoothed potential $V_{\alpha,\varepsilon}$

defined by

$$V_{\alpha,\varepsilon}(x) := \frac{1}{(|x|^2+\varepsilon^2)^{\alpha/2}}, \qquad x \in \mathbb{R}^2,\ \varepsilon \neq 0. \tag{11.2}$$

Unlike problem (11.1), the regularized problem

$$\begin{cases} \ddot{u} = -\dfrac{\alpha u}{(|u|^2+\varepsilon^2)^{(\alpha+2)/2}} \\ \\ (u(0), \dot{u}(0)) = (q_0, p_0) \in \mathbb{R}^2 \times \mathbb{R}^2. \end{cases} \tag{11.3}$$

admits a unique global solution in $C^\infty([0,+\infty[;\mathbb{R}^2)$ for any choice of the initial data.

After this smoothing procedure, natural questions are to investigate the existence of the asymptotic limits as $\varepsilon \to 0$ of the solutions of (11.3), to study their continuity with respect to the initial data and to compare these limits with the original solutions of (11.1). To the best of our knowledge the answers to these questions are not present in the literature; however we recall that the smoothed potential $V_{1,\varepsilon}$ appears in the astrophysical literature (see [17], [46]), in the study of Vlasov's equations for Newtonian interactions (see [61]), and even in classical electrodynamics (see [36]).

By a proper choice of the reference frame, of the initial time and of the standard units we can assume that the initial conditions of (11.1) and (11.3) are

$$\begin{cases} u(0) = q_0 = (1,0) \\ \dot{u}(0) = p_0 = (0,\nu), \end{cases} \tag{11.4}$$

where ν is a real parameter.

Let us denote by $u_{\varepsilon,\nu} \in C^\infty[0,+\infty[;\mathbb{R}^2)$ the solution of (11.3) and by $u_\nu \in C^\infty([0,T_\nu[;\mathbb{R}^2)$ the solution of (11.1), with initial data as in (11.4). To understand better the phenomena that may occur when we consider the limits of $u_{\varepsilon,\nu}$ as $\varepsilon \to 0$, let us consider the Kepler problem $\alpha = 1$: in this case, for $\nu \neq 0$, the function $u_\nu(t) = \lim_{\varepsilon\to 0} u_{\varepsilon,\nu}(t)$ is defined for all the values of $t \in [0,+\infty[$. Therefore, we only need to study the case $\nu = 0$, for which the trajectory $u_0([0,T_0[)$ of u_0, with $T_0 = \pi/(2\sqrt{2})$, is the segment joining the point $(1,0)$ with the origin. We observe that the function $u_{\varepsilon,0}$ is periodic and its trajectory is the segment $[-1,1]\times\{0\}$ joining the point $(1,0)$ with the point $(-1,0)$ (in particular, $u_{\varepsilon,0}$ crosses the origin, that is the collision point). Moreover, there exists the limit function $\lim_{\varepsilon\to 0} u_{\varepsilon,0}(t)$, defined for $t \in [0,+\infty[$ and continuous on its definition domain; it is also smooth out of the set

$$\left\{ (1+2k)\frac{\pi}{2\sqrt{2}} : k \in \mathbb{Z} \right\};$$

it is periodic of period $4T_0$, and its trajectory is still the segment $[-1,1]\times\{0\}$ and is made by patching together pieces of the trajectory of u_0. We shall refer to the function $\lim_{\varepsilon\to 0} u_{\varepsilon,0}(t)$ as to the *transmission solution*.

Note that the transmission solution does *not* coincide with the global solution (that we shall call *reflection solution*, or *collision-ejection solution*) usually considered in the literature, which is the $2T_0$-periodic solution having as trajectory the segment $[0,1] \times \{0\}$ that joins the point $(1,0)$ to the origin. The reflection solution, still denoted by u_0 for simplicity, has been obtained with different methods in several works, see for instance [2], [42], [66], [68], [45], [60].

The reflection solution is continuous with respect to the initial data, while this is *not* the case for the transmission solution. We enhance this consideration with the following statement:

Let $\alpha = 1$. *The function* $\lim_{\varepsilon\to 0} u_{\varepsilon,\nu}(t)$, *defined for all* $t \in [0, +\infty[$, *is not continuous, if considered as a function of* ν, *at* $\nu = 0$*: precisely, we have*

$$\lim_{\nu\to 0}\,\lim_{\varepsilon\to 0} u_{\varepsilon,\nu} \;=\; u_0 \;\neq\; \lim_{\varepsilon\to 0} u_{\varepsilon,0} \;=\; \lim_{\varepsilon\to 0}\lim_{\nu\to 0} u_{\varepsilon,\nu}\,. \tag{11.5}$$

The fact that the two limits in (11.5) do not commute shows clearly that the double limits as $(\varepsilon,\nu) \to (0,0)$ of $u_{\varepsilon,\nu}$ do not exist, at least for $\alpha = 1$. This leads naturally to investigate the existence of the double limits as of u_{ε_k,ν_k} for some subsequence $(\varepsilon_k,\nu_k) \to (0,0)$ and for each $\alpha > 0$.

We recall that in Chapter 2 we have seen that for $\alpha \geq 2$ there are nonzero values of the angular momentum leading to collision: for this reason we expect a different behavior for the previous limits if $\alpha \geq 2$.

Since it results $u_{\varepsilon,\nu}(t) = -u_{\varepsilon,-\nu}(t)$, it is sufficient to restrict our analysis to the cases $\nu \geq 0$. The discussion is based on the classical treatment of the 2-body problem in polar coordinates (r,θ) to show that, as discussed in details in Section 11.2, $r_{\varepsilon,\nu}(t)$ oscillates periodically with period $2T_{\varepsilon,\nu} > 0$ between $r = r^+_{\varepsilon,\nu}$ $(0 < r^+_{\varepsilon,\nu} < 1)$ and $r = 1$ while the increment of the angle between two successive maxima of $r_{\varepsilon,\nu}(t)$ is given by $2\Theta_{\varepsilon,\nu}$.

The pointwise double limit

$$\lim_{(\varepsilon,\nu)\to(0,0)} \Theta_{\varepsilon,\nu} \tag{11.6}$$

does not exist for $0 < \alpha \leq 2$: in fact we can find different limit values of $\Theta_{\varepsilon,\nu}$ by restricting to particular sequences (ε_k,ν_k) $(k \in \mathbb{N})$, as $k \to \infty$. In particular if $0 < \alpha < 2$ we can find a bounce as limit trajectory with every angular value between $\pi/2$ and $\pi/(2-\alpha)$; if $\alpha = 2$ we can find a bounce as limit trajectory with every angle between $\pi/2$ and $+\infty$.

We find a different behavior for $\alpha > 2$: in this case there exists the limit of $\Theta_{\varepsilon,\nu}$ for $(\varepsilon,\nu) \to (0,0)$ and its value is $\pi/2$.

From the study of these limits of $\Theta_{\varepsilon,\nu}$ we can deduce the limit behavior of the trajectories of the problem. We plot in Figure 11.1 these trajectories in the most interesting case: $0 < \alpha < 2$.

The content of this chapter is based on [3].

11.2 Preliminaries

Let $\alpha > 0$. We are interested in finding the limit trajectories, corresponding to a collision solution, when we perturb the equation (11.1) by replacing V_α with the regularized

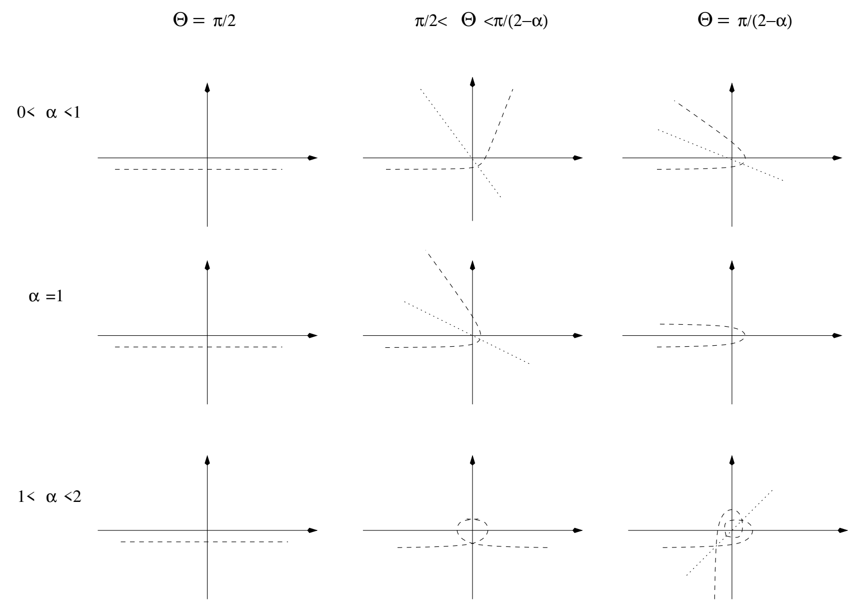

Figure 11.1: This figure shows the results of Theorem 19: we plot trajectories of the solutions $u_{\varepsilon,\nu}$ for (ε,ν) close to $(0,0)$ and $0<\alpha<2$.

potential $V_{\alpha,\varepsilon}$,

$$V_{\alpha,\varepsilon}(x)=\frac{1}{\left(|x|^2+\varepsilon^2\right)^{\alpha/2}},\qquad x\in\mathbb{R}^2,\ \varepsilon\neq 0, \tag{11.7}$$

and, at the same time, we perturb the initial data (q_0,p_0).

Let $|\nu|=|q_0\wedge p_0|$ be the (rescaled) angular momentum of the system. We shall assume that $|\nu|$ is small enough, in order to ensure the system to have negative energy.

The proof of the following lemma is straightforward.

LEMMA 4. *Let $\alpha>0$, $\varepsilon\neq 0$, and let $(q,p)\in\mathbb{R}^4$. Then there exists a unique global solution in $C^\infty([0,+\infty[;\mathbb{R}^2)$ of the regularized problem (11.3).*

Proof. Problem (11.3) is equivalent to

$$\begin{cases} \dot u = v, \\ \dot v = -\dfrac{\alpha u}{\left(|u|^2+\varepsilon^2\right)^{(\alpha+2)/2}}, \quad t\in\,]0,+\infty[, \\ (u(0),v(0))=(q_0,p_0). \end{cases} \tag{11.8}$$

The right hand side of the first two equations in (11.8) belongs to $\mathrm{Lip}(\mathbb{R}^4;\mathbb{R}^4)\cap C^\infty(\mathbb{R}^4;\mathbb{R}^4)$, and the statement follows. □

REMARK 32. The analogous of Lemma 4 holds also in the case of the regularized N-body problem.

In the sequel we assume, without loss of generality, that $q_0 = (1,0)$ and $p_0 = (0,\nu)$ in (11.1) and (11.3).

The regularized system (11.3) with initial condition as in (11.4) admits the two classical first integrals of the energy and of the angular momentum:

$$\frac{1}{2}|\dot{u}|^2 - \frac{1}{(u^2+\varepsilon^2)^{\alpha/2}} = E, \qquad u \wedge \dot{u} = \nu e_3$$

where e_3 is the unit vector perpendicular to the plane of the motion.

If we use polar coordinates (r,θ), we can write the energy and the norm of the angular momentum as

$$\begin{cases} \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - V_{\alpha,\varepsilon}(r) = \frac{1}{2}\nu^2 - V_{\alpha,\varepsilon}(1), \\ \\ r^2\dot{\theta} = \nu. \end{cases} \tag{11.9}$$

Let us set $r_\theta := dr/d\theta$, we obtain

$$r_\theta^2 = \dot{r}^2\dot{\theta}^{-2} = \dot{r}^2\frac{r^4}{\nu^2} = r^2 g_{\varepsilon,\nu}(r),$$

where

$$\begin{aligned} g_{\varepsilon,\nu}(r) \quad &:= \dot{r}^2\frac{r^2}{\nu^2} = \frac{r^2}{\nu^2}\left[-r^2\dot{\theta}^2 + 2\Big(V_{\alpha,\varepsilon}(r) - V_{\alpha,\varepsilon}(1) + \frac{1}{2}\nu^2\Big)\right] = \\ &= \left[1 + \frac{2}{\nu^2}\Big(V_{\alpha,\varepsilon}(r) - V_{\alpha,\varepsilon}(1)\Big)\right]r^2 - 1, \qquad r \in [0,1]. \end{aligned} \tag{11.10}$$

For simplicity, we still denote $2/\nu^2$ by $1/\nu^2$, so that (11.10) reads as

$$g_{\varepsilon,\nu}(r) := \left[1 + \frac{1}{\nu^2}\Big(V_{\alpha,\varepsilon}(r) - V_{\alpha,\varepsilon}(1)\Big)\right]r^2 - 1, \qquad r \in [0,1]. \tag{11.11}$$

Substituting the expression (11.7) of $V_{\alpha,\varepsilon}$ in (11.11) we obtain

$$g_{\varepsilon,\nu}(r) = \frac{\nu^2\left(r^2+\varepsilon^2\right)^{\alpha/2}\left[r^2-1\right] + \left[1 - \left(\dfrac{r^2+\varepsilon^2}{1+\varepsilon^2}\right)^{\alpha/2}\right]r^2}{\nu^2(r^2+\varepsilon^2)^{\alpha/2}}\,. \tag{11.12}$$

The whole trajectory can be obtained by symmetry from a part of the orbit joining a point where r is maximum (in this case equal to 1) and a point where r is minimum (see [82]). We call $r_+ = r_+(\varepsilon,\nu,\alpha)$ the minimal value of r: as shown in Proposition 11, r_+ is the only positive root of $g_{\varepsilon,\nu}(r) = 0$ besides $r = 1$ and we shall see that it is infinitesimal with ε,ν. In Proposition 11 of Section 11.3 we shall also estimate the rate of convergence of $r_+(\varepsilon,\nu,\alpha)$ to zero as $(\varepsilon,\nu) \to (0,0)$.

While r varies from 1 to r_+, the angle θ increases of the quantity

$$\Theta_{\varepsilon,\nu} := \int_{r_+}^{1} \frac{1}{r\sqrt{g_{\varepsilon,\nu}(r)}}\,dr. \tag{11.13}$$

The behavior of $\Theta_{\varepsilon,\nu}$ as $(\varepsilon,\nu) \to (0,0)$ will be crucial in understanding the limit trajectories.

REMARK 33. The integrand function in (11.13) tends to zero as $(\varepsilon,\nu) \to (0,0)$; indeed $g_{\varepsilon,\nu}(r) = h_{\varepsilon,\nu}(r)/\nu^2$ and

$$\lim_{(\varepsilon,\nu)\to(0,0)} h_{\varepsilon,\nu}(r) = (1-r^\alpha)r^{2-\alpha}, \qquad r > 0.$$

In the following we shall study the order of infinitesimal of r_+: we shall see that there is a coordinate change transforming the integrand in (11.13) into a sequence of functions that are not infinitesimal with ε,ν.

11.3 The order of infinitesimal of $r_+(\varepsilon,\nu,\alpha)$

LEMMA 5. *Let $\alpha > 0$ and denote by g the function $g_{\varepsilon,\nu}$ defined in (11.11). Then there exists $\nu_\alpha > 0$ such that for any $\nu \in]0,\nu_\alpha[$ and any $\varepsilon \in]0,1[$ we can find $r_+ = r_+(\varepsilon,\nu,\alpha)$ satisfying the following properties:*

$$r_+ \in]0,1[, \qquad g(r_+) = 0, \qquad r \in]r_+,1[\Longrightarrow g(r) > 0. \tag{11.14}$$

Moreover

$$\lim_{\nu\to 0} r_+(\varepsilon,\nu,\alpha) = 0. \tag{11.15}$$

Proof. From (11.11) it follows

$$g(0) = -1, \qquad g(1) = 0, \tag{11.16}$$

and

$$g'(r) = 2r\left[1 + \frac{1}{\nu^2}\left(\frac{1}{(r^2+\varepsilon^2)^{\alpha/2}} - \frac{1}{(1+\varepsilon^2)^{\alpha/2}}\right)\right] - \frac{\alpha}{\nu^2}\frac{r^3}{(r^2+\varepsilon^2)^{(\alpha+2)/2}}. \tag{11.17}$$

Hence

$$g'(1) = 2 - \frac{\alpha}{\nu^2}\frac{1}{(1+\varepsilon^2)^{(\alpha+2)/2}} \leq 2 - \frac{\alpha}{2^{(\alpha+2)/2}\nu^2} \qquad \forall \varepsilon \in]0,1[.$$

Therefore, if we set

$$\nu_\alpha := \frac{\alpha^{1/2}}{2^{(\alpha+4)/4}}$$

we have
$$g'(1) < 0, \qquad \nu \in \,]0,\nu_\alpha[\,. \tag{11.18}$$

From (11.16) and (11.18) it follows that there is some point of $]0,1[$ where g vanishes. In order to obtain r_+ as in (11.14) it is then enough to define
$$r_+ := \sup\{r \in \,]0,1[\,: g(r) = 0\}\,.$$

By definition, r_+ is a solution of the system $g(r) = 0$ and $g'(r) \geq 0$. These two conditions imply
$$0 \leq r_+ g'(r_+) = 2[g(r_+)+1] - \frac{\alpha}{\nu^2}\frac{r_+^4}{(r_+^2+\varepsilon^2)^{(\alpha+2)/2}} = 2 - \frac{\alpha}{\nu^2}\frac{r_+^4}{(r_+^2+\varepsilon^2)^{(\alpha+2)/2}}$$
provided $\varepsilon \in \,]0,1[$. Therefore
$$r_+^4 \leq \frac{2\nu^2}{\alpha}(r_+^2+\varepsilon^2)^{(\alpha+2)/2} \leq \frac{2^{(\alpha+4)/2}\nu^2}{\alpha}, \tag{11.19}$$
and (11.15) follows. □

Proposition 11. *Let r_+ be as in Lemma 5. Then the following holds:*
$$\alpha \geq 2 \Longrightarrow \lim_{(\varepsilon,\nu)\to(0,0)} \frac{r_+}{\nu\varepsilon^{\alpha/2}} = 1; \tag{11.20}$$
$$0 < \alpha < 2 \text{ and } \lim_{(\varepsilon,\nu)\to(0,0)} \frac{\varepsilon^{2-\alpha}}{\nu^2} = +\infty \Longrightarrow \lim_{(\varepsilon,\nu)\to(0,0)} \frac{r_+}{\nu\varepsilon^{\alpha/2}} = 1; \tag{11.21}$$
$$0 < \alpha < 2 \text{ and } \lim_{(\varepsilon,\nu)\to(0,0)} \frac{\varepsilon^{2-\alpha}}{\nu^2} = \delta^2 \Longrightarrow \lim_{(\varepsilon,\nu)\to(0,0)} \frac{r_+}{\nu^{2/(2-\alpha)}} = \gamma(\delta), \tag{11.22}$$
where $\gamma^2(\delta)$ is the unique positive solution of the equation
$$x + \delta^{4/(2-\alpha)} = x^{2/\alpha}\,; \tag{11.23}$$
$$0 < \alpha < 2 \text{ and } \lim_{(\varepsilon,\nu)\to(0,0)} \frac{\varepsilon^{2-\alpha}}{\nu^2} = 0 \Longrightarrow \lim_{(\varepsilon,\nu)\to(0,0)} \frac{r_+}{\nu^{2/(2-\alpha)}} = 1. \tag{11.24}$$

Proof. Since $r_+ > 0$, the equation $g(r_+) = 0$ is equivalent to
$$\nu^2\frac{(r_+^2+\varepsilon^2)^{\alpha/2}}{r_+^2}(r_+^2-1) + \left[1 - \left(\frac{r_+^2+\varepsilon^2}{1+\varepsilon^2}\right)^{\alpha/2}\right] = 0. \tag{11.25}$$

Taking the limit of (11.25) as $(\varepsilon,\nu) \to (0,0)$ yields
$$\lim_{(\varepsilon,\nu)\to(0,0)} \nu^2\varepsilon^{\alpha-2}\frac{(r_+^2/\varepsilon^2+1)^{\alpha/2}}{r_+^2/\varepsilon^2} = 1. \tag{11.26}$$

If $0<\alpha<2$ and $\lim_{(\varepsilon,\nu)\to(0,0)}\nu^2\varepsilon^{\alpha-2}=0$, then relation (11.26) implies $\lim_{(\varepsilon,\nu)\to(0,0)} r_+^2/\varepsilon^2=0$; therefore, still using (11.26), we obtain $\lim_{(\varepsilon,\nu)\to(0,0)}\nu^2\varepsilon^{\alpha}/r_+^2=1$, and this proves (11.21).

If $\alpha=2$ we always have $\lim_{(\varepsilon,\nu)\to(0,0)}\nu^2\varepsilon^{\alpha-2}=0$, and also in this case relation (11.26) implies $\lim_{(\varepsilon,\nu)\to(0,0)} r_+^2/\varepsilon^2=0$. In the case $\alpha\geq 2$, even if we always have $\lim_{(\varepsilon,\nu)\to(0,0)}\nu^2\varepsilon^{\alpha-2}=0$, then relation (11.26) implies either $\lim_{(\varepsilon,\nu)\to(0,0)} r_+^2/\varepsilon^2=0$ or $\lim_{(\varepsilon,\nu)\to(0,0)} r_+^2/\varepsilon^2=+\infty$. We want to exclude the second possibility: if, by contradiction, $\lim_{(\varepsilon,\nu)\to(0,0)} r_+^2/\varepsilon^2=+\infty$ then by (11.19) we would have

$$\frac{r_+^4}{\varepsilon^4}\leq\frac{2\nu^2}{\alpha}\varepsilon^{\alpha-2}\left(\frac{r_+^2}{\varepsilon^2}+1\right)^{(\alpha+2)/2}<C\nu^2\varepsilon^{\alpha-2}\left(\frac{r_+}{\varepsilon}\right)^{\alpha+2}$$

for ε,ν small enough, and for some positive constant C. This implies $r_+\to+\infty$ for $(\varepsilon,\nu)\to(0,0)$. Then we can state that if $\alpha\geq 2$ we always have $\lim_{(\varepsilon,\nu)\to(0,0)}\frac{r_+}{\nu\varepsilon^{\alpha/2}}=1$. The proof of relation (11.20) is then immediate, by following the same steps done for (11.21).

If $0<\alpha<2$ and $\lim_{(\varepsilon,\nu)\to(0,0)}\nu^2\varepsilon^{\alpha-2}=1/\delta^2$ for some $\delta>0$, then (11.26) implies $\lim_{(\varepsilon,\nu)\to(0,0)} r_+^2/\nu^{4/(2-\alpha)}=\gamma^2(\delta)$, where $\gamma^2(\delta)$ is the unique strictly positive solution of the equation $x+\delta^{4/(2-\alpha)}=x^{2/\alpha}$; the existence of $\lim_{(\varepsilon,\nu)\to(0,0)} r_+^2/\nu^{4/(2-\alpha)}$ is a direct consequence of the existence of $\lim_{(\varepsilon,\nu)\to(0,0)} r_+^2/\varepsilon^2$, which follows from the monotonicity of the function $x\to(x+1)^{\alpha/2}/x$, while the uniqueness of $\gamma^2(\delta)$ is a direct consequence of the first two of the following properties:

1. the function $\delta\in\,]0,+\infty[\,\to\gamma(\delta)$ is strictly increasing;
2. $\lim_{\delta\to 0^+}\gamma(\delta)=1$;
3. $\lim_{\delta\to+\infty}\gamma(\delta)=+\infty$.

The proof of these properties can be obtained using the implicit function theorem and equation (11.23).

Finally, if $0<\alpha<2$ and $\lim_{(\varepsilon,\nu)\to(0,0)}\nu^2\varepsilon^{\alpha-2}=+\infty$, then (11.26) implies $\lim_{(\varepsilon,\nu)\to(0,0)} r_+^2/\varepsilon^2=+\infty$; therefore $\lim_{(\varepsilon,\nu)\to(0,0)}\nu^2 r_+^{\alpha-2}=1$, which is equivalent to (11.24). □

Remark 34. By Proposition 11 we immediately have that

$$\lim_{(\varepsilon,\nu)\to(0,0)}\frac{r_+^2}{\nu^2}=0$$

for any $\alpha>0$.

Corollary 2. *There exists $a_\alpha>0$ such that*

$$(\varepsilon,\nu)\in\,]0,a_\alpha[\,^2\Longrightarrow g'(r_+)>0. \tag{11.27}$$

Proof. To prove (11.27) we study the sign of $(1/2)r_+g'(r_+)$. Recalling (11.17) we have

$$\frac{1}{2}r_+g'(r_+) = r_+^2 + \frac{r_+^2}{\nu^2(r_+^2+\varepsilon^2)^{\alpha/2}} - \frac{r_+^2}{\nu^2(1+\varepsilon^2)^{\alpha/2}} - \frac{\alpha}{2}\frac{r_+^2}{\nu^2(r_+^2+\varepsilon^2)^{\alpha/2}}\frac{r_+^2}{(r_+^2+\varepsilon^2)}. \tag{11.28}$$

Taking the limit of (11.28) as $(\varepsilon,\nu)\to(0,0)$ and recalling (11.15), (11.26), and Remark 34, we get

$$\lim_{(\varepsilon,\nu)\to(0,0)}\frac{1}{2}r_+g'(r_+) = 1-\frac{\alpha}{2}\lim_{(\varepsilon,\nu)\to(0,0)}\frac{\left(\frac{r_+}{\varepsilon}\right)^2}{1+\left(\frac{r_+}{\varepsilon}\right)^2}. \tag{11.29}$$

We now observe that the right hand side of (11.29) is strictly positive for ε and ν small enough. Indeed, if $\alpha\ge 2$ we have $\lim_{(\varepsilon,\nu)\to(0,0)} r_+/\varepsilon = 0$. On the other hand, if $0<\alpha<2$, then $\lim_{(\varepsilon,\nu)\to(0,0)} r_+g'(r_+)/2 \ge 1-\alpha/2 > 0$. □

11.4 The variable change and the boundedness of $1/K_{\varepsilon,\nu,\alpha}$

The numerator of (11.12) is an even function of the variable r, so that it vanishes also in $r_- := -r_+$. We extract the three roots $r_-, r_+, 1$ from $g_{\varepsilon,\nu}(r)$ in the following way:

$$\Theta_{\varepsilon,\nu} = \int_{r_+}^{1}\frac{\nu(r^2+\varepsilon^2)^{\alpha/4}}{r\sqrt{(r^2-r_+^2)(1-r)K_{\varepsilon,\nu,\alpha}(r)}}\,dr, \tag{11.30}$$

where

$$K_{\varepsilon,\nu,\alpha}(r) := \frac{\nu^2\left(r^2+\varepsilon^2\right)^{\alpha/2}\left[r^2-1\right]+\left[1-\left(\frac{r^2+\varepsilon^2}{1+\varepsilon^2}\right)^{\alpha/2}\right]r^2}{(r^2-r_+^2)(1-r)}. \tag{11.31}$$

Setting

$$r = r(\rho) := r_+\rho \tag{11.32}$$

we obtain

$$\Theta_{\varepsilon,\nu} = \int_1^{1/r_+} f_{\varepsilon,\nu}(\rho)\,d\rho, \tag{11.33}$$

with

$$f_{\varepsilon,\nu}(\rho) := \frac{\nu(r_+^2\rho^2+\varepsilon^2)^{\alpha/4}}{r_+\rho\sqrt{(\rho^2-1)(1-r_+\rho)K_{\varepsilon,\nu,\alpha}(r_+\rho)}}. \tag{11.34}$$

We now give a uniform bound on the function $1/K_{\varepsilon,\nu,\alpha}$ defined in (11.31): more precisely, it turns out to be easier to give a bound on $(1+r)/K_{\varepsilon,\nu,\alpha}(r)$, which is a function of the variable r^2. This result, as well as Proposition 13 below, will be necessary to prove the main results in Section 11.5. Recall that

$$\frac{1+r}{K_{\varepsilon,\nu,\alpha}(r)} = \frac{(1-r^2)(r^2-r_+^2)}{\nu^2(r^2+\varepsilon^2)^{\alpha/2}[r^2-1] + \left[1-\left(\frac{r^2+\varepsilon^2}{1+\varepsilon^2}\right)^{\alpha/2}\right] r^2} . \tag{11.35}$$

PROPOSITION 12. *For each* $\alpha > 0$ *there exist* $L_\alpha > 0$ *and* $a_\alpha > 0$ *such that*

$$\Psi(r) := \frac{1+r}{K_{\varepsilon,\nu,\alpha}(r)} < L_\alpha, \qquad r \in [r_+, 1],\ (\varepsilon,\nu) \in]0, a_\alpha[^2 .$$

Proof. We divide the numerator and the denominator of (11.35) by $r_+^2(1-r^2)$ and we apply the change of variables $r = r_+\rho$, so that

$$\Psi(r_+\rho) = \frac{\rho^2-1}{P(r_+^2\rho^2)\,(\rho^2-Q(\rho^2))}\,, \qquad 1 \le \rho^2 \le \frac{1}{r_+^2}, \tag{11.36}$$

where

$$\begin{aligned} P(s) &:= \frac{1-\left(\frac{s+\varepsilon^2}{1+\varepsilon^2}\right)^{\alpha/2}}{1-s}\,, \qquad r_+^2 \le s \le 1 \\ Q(\rho^2) &:= \frac{\nu^2(r_+^2\rho^2+\varepsilon^2)^{\alpha/2}}{r_+^2 P(r_+^2\rho^2)}, \qquad 1 \le \rho^2 \le \frac{1}{r_+^2}. \end{aligned} \tag{11.37}$$

First we observe that there exist constants $\kappa_\alpha, K_\alpha > 0$ such that

$$\kappa_\alpha < P(s) < K_\alpha, \qquad 0 \le s \le 1,\ \varepsilon \in [0,1] .$$

Hence the boundedness of $\Psi(r)$ is equivalent to the boundedness of the function

$$\frac{\rho^2-1}{\rho^2-Q(\rho^2)}, \qquad 1 \le \rho^2 \le \frac{1}{r_+^2}. \tag{11.38}$$

The definition of r_+ as a root of the equation $g_{\varepsilon,\nu} = 0$, $g_{\varepsilon,\nu}$ as in (11.12), is equivalent to $Q(1) = 1$. hence

$$\frac{\rho^2-1}{\rho^2-Q(\rho^2)} = \frac{1}{1-\int_0^1 Q'(1+s(\rho^2-1))\,ds}. \tag{11.39}$$

Therefore, to prove that the function in (11.38) is bounded, we need to prove a bound on $Q'(\rho^2)$ for $\rho^2 \in [1, 1/r_+^2]$. We compute

$$Q'(\rho^2) = \nu^2\left(r_+^2\rho^2+\varepsilon^2\right)^{(\alpha-2)/2}\frac{\alpha}{2}\frac{[1-w(r_+^2\rho^2)]}{P(r_+^2\rho^2)} \tag{11.40}$$

where

$$w(r_+^2\rho^2) := \frac{2}{\alpha}\frac{(r_+^2\rho^2+\varepsilon^2)}{(1-r_+^2\rho^2)} - \left(\frac{r_+^2\rho^2+\varepsilon^2}{1+\varepsilon^2}\right)^{\alpha/2} \frac{1}{1-\left(\frac{r_+^2\rho^2+\varepsilon^2}{1+\varepsilon^2}\right)^{\alpha/2}} .$$

LEMMA 6. *For each $\alpha > 0$ there exist $\eta_\alpha \in]0,1[$, $c_\alpha \in]0,1[$ and $a_\alpha > 0$ such that*

$$Q'(\rho^2) < 1-\eta_\alpha, \qquad 1 \le \rho^2 \le \frac{c_\alpha}{r_+^2}, \qquad (\varepsilon,\nu) \in]0,a_\alpha[^2 .$$

Proof. Let $d \in]0,1/2[$ be a constant; for $1 \le \rho^2 \le d/r_+^2$ and $\varepsilon^2 < d/(1-2d)$ we have

$$\frac{1}{P(r_+^2\rho^2)} = \frac{1-r_+^2\rho^2}{1-\left(\frac{r_+^2\rho^2+\varepsilon^2}{1+\varepsilon^2}\right)^{\alpha/2}} \le \frac{1}{1-\left(\frac{r_+^2\rho^2+\varepsilon^2}{1+\varepsilon^2}\right)^{\alpha/2}} < \frac{1}{1-(2d)^{\alpha/2}}$$

$$-w(r_+^2\rho^2) < \left(\frac{r_+^2\rho^2+\varepsilon^2}{1+\varepsilon^2}\right)^{\alpha/2} \frac{1}{1-\left(\frac{r_+^2\rho^2+\varepsilon^2}{1+\varepsilon^2}\right)^{\alpha/2}} < \frac{(2d)^{\alpha/2}}{1-(2d)^{\alpha/2}}$$

so that

$$\frac{\alpha}{2}\frac{[1-w(r_+^2\rho^2)]}{P(r_+^2\rho^2)} < \frac{\alpha}{2}\frac{1}{1-(2d)^{\alpha/2}}\left(1+\frac{(2d)^{\alpha/2}}{1-(2d)^{\alpha/2}}\right) . \tag{11.41}$$

If $\alpha \ge 2$ we choose

$$\eta_\alpha := \frac{1}{4}, \qquad c_\alpha := \frac{1}{4}, \qquad \widetilde{a}_\alpha := \sqrt{\frac{c_\alpha}{1-2c_\alpha}} .$$

If $0 < \alpha < 2$ we choose

$$\eta_\alpha := \left(\frac{2-\alpha}{4}\right)^2 ,$$

we take $c_\alpha \in]0,\frac{1}{2}[$ small enough in such a way that

$$\frac{\alpha}{2}\frac{1}{1-(2c_\alpha)^{\alpha/2}}\left(1+\frac{(2c_\alpha)^{\alpha/2}}{1-(2c_\alpha)^{\alpha/2}}\right) < 1-\frac{2-\alpha}{4},$$

and we set $\widetilde{a}_\alpha := \sqrt{(c_\alpha)/(1-2c_\alpha)}$. With this definition of c_α, applying (11.41) with c_α in place of d, we get

$$\frac{\alpha}{2}\frac{[1-w(r_+^2\rho^2)]}{P(r_+^2\rho^2)} < \begin{cases} \frac{\alpha}{2}\frac{1}{1-(1/2^{\alpha/2})}\left(1+\frac{1/2^{\alpha/2}}{1-1/2^{\alpha/2}}\right) = \frac{\alpha}{2}\frac{2^\alpha}{(2^{\alpha/2}-1)^2} & \text{if } \alpha \ge 2 \\ 1-\frac{2-\alpha}{4} & \text{if } 0 < \alpha < 2 , \end{cases} \tag{11.42}$$

provided $1 \le \rho^2 \le c_\alpha / r_+^2$ and $\varepsilon \in]0, \widetilde{a}_\alpha[$.

For $\alpha \ge 2$ we set

$$\xi_\alpha := \frac{\alpha}{2} \frac{2^\alpha}{(2^{\alpha/2}-1)^2}$$

If $\alpha \ge 2$ we have

$$\nu^2 (r_+^2 \rho^2 + \varepsilon^2)^{(\alpha-2)/2} \le \nu^2 (c_\alpha + \widetilde{a}_\alpha^2)^{(\alpha-2)/2}, \qquad 1 \le \rho^2 \le \frac{c_\alpha}{r_+^2}, \ \varepsilon \in]0, \widetilde{a}_\alpha[. \tag{11.43}$$

For $\alpha \ge 2$, if we define

$$a_\alpha := \min \left(\widetilde{a}_\alpha, \sqrt{\frac{(c_\alpha + \widetilde{a}_\alpha^2)^{(2-\alpha)/2}}{2\xi_\alpha}} \right),$$

from (11.40), (11.42) and (11.43) we deduce that

$$Q'(\rho^2) = \nu^2 (c_\alpha + \tilde{a}_\alpha^2)^{(\alpha-2)/2} \frac{\alpha}{2} \frac{2^\alpha}{(2^{\alpha/2}-1)^2} \le \frac{1}{2}, \qquad 1 \le \rho^2 \le \frac{c_\alpha}{r_+^2}, \quad (\varepsilon, \nu) \in]0, a_\alpha[. \tag{11.44}$$

For $0 < \alpha < 2$ we have

$$\nu^2 (r_+^2 \rho^2 + \varepsilon^2)^{(\alpha-2)/2} \le \frac{1}{\left(\frac{\varepsilon^2}{\nu^{4/(2-\alpha)}} + \frac{r_+^2}{\nu^{4/(2-\alpha)}} \right)^{(2-\alpha)/2}} \qquad \forall \rho \ge 1 .$$

We observe that there is $a_\alpha \in]0, \widetilde{a}_\alpha[$ such that $(\varepsilon, \nu) \in]0, a_\alpha[^2$ implies

$$\frac{1}{\left(\frac{\varepsilon^2}{\nu^{4/(2-\alpha)}} + \frac{r_+^2}{\nu^{4/(2-\alpha)}} \right)^{(2-\alpha)/2}} < 1 + \frac{2-\alpha}{4} . \tag{11.45}$$

Indeed, if $\lim_{(\varepsilon,\nu)\to(0,0)} \nu^2 \varepsilon^{\alpha-2} = 0$ then (11.45) easily follows while, if $\lim_{(\varepsilon,\nu)\to(0,0)} \nu^2 \varepsilon^{\alpha-2} = +\infty$, then (11.45) follows from (11.24).

Assume now that $\lim_{(\varepsilon,\nu)\to(0,0)} \nu^2 \varepsilon^{\alpha-2} = 1/\delta^2$ for some $\delta \in]0, +\infty[$, and assume by contradiction that (11.45) is false. Then there is a sequence (ε_n, ν_n) converging to $(0,0)$ such that, setting $r_n := r_+(\varepsilon_n, \nu_n, \alpha)$, we have

$$\frac{1}{\left(\frac{\varepsilon_n^2}{\nu_n^{4/(2-\alpha)}} + \frac{r_n^2}{\nu_n^{4/(2-\alpha)}} \right)^{(2-\alpha)/2}} \ge 1 + \frac{2-\alpha}{4}, \qquad n \in \mathbb{N}. \tag{11.46}$$

Passing to the limit for $n \to \infty$ and using relation (11.22) we obtain

$$\lim_{n\to\infty} \frac{1}{\left(\frac{\varepsilon_n^2}{\nu_n^{4/(2-\alpha)}} + \frac{r_n^2}{\nu_n^{4/(2-\alpha)}} \right)^{(2-\alpha)/2}} = \frac{1}{\left[\delta^{4/(2-\alpha)} + \gamma^2(\delta) \right]^{(2-\alpha)/2}} \le 1, \tag{11.47}$$

where the last inequality in (11.47) follows from $\gamma^2(\delta)+\delta^{\frac{4}{2-\alpha}} = \gamma^2(\delta) \geq 1$. Then (11.47) contradicts (11.46), and therefore (11.45) holds.

Then from (11.40), (11.42) and (11.45) we deduce that, for $0<\alpha<2$, we have

$$Q'(\rho^2) \leq \frac{1}{2}, \qquad 1 \leq \rho^2 \leq \frac{c_\alpha}{r_+^2}, \qquad (\varepsilon,\nu) \in \,]0,a_\alpha[. \tag{11.48}$$

The proof of the Lemma then follows from (11.44) and (11.48).

□

From the Lemma and (11.39) it follows that

$$\frac{\rho^2-1}{\rho^2-Q(\rho^2)} < \frac{1}{\eta_\alpha} \qquad \text{if} \quad 1 \leq \rho^2 < \frac{c_\alpha}{r_+^2}, \qquad (\varepsilon,\nu) \in \,]0,a_\alpha[. \tag{11.49}$$

On the other hand for $1 \leq \rho^2 < 1/r_+^2$ and $(\varepsilon,\nu) \in]0,a_\alpha[^2$ we have, possibly reducing $a_\alpha > 0$,

$$Q(\rho^2) < \frac{a_\alpha^2}{r_+^2}\frac{(1+a_\alpha^2)^{\alpha/2}}{\kappa_\alpha} < \frac{c_\alpha}{4r_+^2}\,.$$

It follows that

$$\frac{\rho^2-1}{\rho^2-Q(\rho^2)} < \frac{\rho^2-1}{\rho^2-\frac{c_\alpha}{4r_+^2}} \leq 2 \qquad \text{if} \quad \rho^2 \geq \frac{c_\alpha}{2r_+^2}\,. \tag{11.50}$$

From (11.49) and (11.50) we can conclude that

$$\Psi(r) < \frac{1}{\kappa_\alpha}\max\left\{2,\frac{1}{\eta_\alpha}\right\} \qquad r \in [r_+,1], \quad \alpha>0, \quad (\varepsilon,\nu) \in]0,a_\alpha[^2\,.$$

□

We shall also use this result:

Proposition 13. *The following relations hold for each $\rho \geq 1$:*

(i) *let* $\alpha \geq 2$, *then* $\displaystyle\lim_{(\varepsilon,\nu)\to(0,0)} \frac{1}{K_{\varepsilon,\nu,\alpha}(r_+\rho)} = 1,$

(ii) *let* $0<\alpha<2$, *then*

$$\begin{cases} \displaystyle\lim_{(\varepsilon,\nu)\to(0,0)} \frac{\varepsilon^{2-\alpha}}{\nu^2} = +\infty \implies \lim_{(\varepsilon,\nu)\to(0,0)} \frac{1}{K_{\varepsilon,\nu,\alpha}(r_+\rho)} = 1\,, \\ \displaystyle\lim_{(\varepsilon,\nu)\to(0,0)} \frac{\varepsilon^{2-\alpha}}{\nu^2} = \delta^2 \implies \lim_{(\varepsilon,\nu)\to(0,0)} \frac{1}{K_{\varepsilon,\nu,\alpha}(r_+\rho)} = \frac{(\rho^2-1)\gamma^2(\delta)}{\rho^2\gamma^2(\delta)-[\rho^2\gamma^2(\delta)+\delta^{4/(2-\alpha)}]^{\alpha/2}}\,, \\ \displaystyle\lim_{(\varepsilon,\nu)\to(0,0)} \frac{\varepsilon^{2-\alpha}}{\nu^2} = 0 \implies \lim_{(\varepsilon,\nu)\to(0,0)} \frac{1}{K_{\varepsilon,\nu,\alpha}(r_+\rho)} = \frac{\rho^2-1}{\rho^2-\rho^\alpha}\,. \end{cases}$$

Proof. Recalling (11.37) we find

$$\lim_{(\varepsilon,\nu)\to(0,0)} P(r_+^2\rho^2) = 1 \qquad \rho \geq 1, \quad \alpha > 0\,. \tag{11.51}$$

Therefore, recalling (11.36), to prove assertions (i) and (ii) we only need to study the pointwise limit of the function $(\rho^2-1)/(\rho^2 - Q(\rho^2))$. Using (11.37) and (11.51) we obtain that

$$\lim_{(\varepsilon,\nu)\to(0,0)} Q(\rho^2) = \lim_{(\varepsilon,\nu)\to(0,0)} \frac{\nu^2(r_+^2\rho^2+\varepsilon^2)^{\alpha/2}}{r_+^2} = \lim_{(\varepsilon,\nu)\to(0,0)} \left(\frac{\frac{r_+^2}{\varepsilon^2}\rho^2+1}{\frac{r_+^2}{\varepsilon^2}+1}\right)^{\alpha/2}, \tag{11.52}$$

where the last equality follows from (11.26) and we shall see that the above limits exist. If $\alpha \geq 2$, from (11.20) we have that $r_+/\varepsilon \to 0$, hence the last limit in (11.52) exists and its value is 1. Therefore

$$\alpha \geq 2, \rho \geq 1 \Longrightarrow \lim_{(\varepsilon,\nu)\to(0,0)} \frac{1}{K_{\varepsilon,\nu,\alpha}(r_+\rho)} = \lim_{(\varepsilon,\nu)\to(0,0)} \frac{\rho^2-1}{\rho^2 - Q(\rho^2)} = 1,$$

and assertion (i) is proved.

Assume now that $0 < \alpha < 2$. If $\lim_{(\varepsilon,\nu)\to(0,0)} \nu^2\varepsilon^{\alpha-2} = 0$, the same reasoning applies, recalling (11.21).

The last case in (ii) follows immediately from (11.52), in fact from (11.24) we get

$$\lim_{(\varepsilon,\nu)\to(0,0)} \frac{\varepsilon^{2-\alpha}}{\nu^2} = 0 \Longrightarrow \lim_{(\varepsilon,\nu)\to(0,0)} \frac{r_+}{\varepsilon} = +\infty\,,$$

and therefore $\lim_{(\varepsilon,\nu)\to(0,0)} Q(\rho^2) = \rho^\alpha$.

If $\lim_{(\varepsilon,\nu)\to(0,0)} \varepsilon^{2-\alpha}/\nu^2 = \delta^2$ for some $\delta \in]0,+\infty[$, we have, by Proposition (11) and the Remark following it,

$$\lim_{(\varepsilon,\nu)\to(0,0)} \frac{r_+^2}{\varepsilon^2} = \frac{\gamma^2(\delta)}{\delta^{4/(2-\alpha)}}\,,$$

therefore

$$\lim_{(\varepsilon,\nu)\to(0,0)} Q(\rho^2) = \left(\frac{\gamma^2(\delta)\rho^2+\delta^{4/(2-\alpha)}}{\gamma^2(\delta)+\delta^{4/(2-\alpha)}}\right)^{\alpha/2} = \frac{\left(\gamma^2(\delta)\rho^2+\delta^{4/(2-\alpha)}\right)^{\alpha/2}}{\gamma^2(\delta)}$$

where we have used the characterization of $\gamma^2(\delta)$ as the solution of the equation

$$x + \delta^{4/(2-\alpha)} = x^{2/\alpha}\,.$$

The proof is concluded. $\square$

11.5 Main results

THEOREM 19. *Let* $0 < \alpha < 2$. *The following assertions hold:*

(i) If $\lim_{(\varepsilon,\nu)\to(0,0)} \dfrac{\varepsilon^{2-\alpha}}{\nu^2} = +\infty$ *then* $\displaystyle\lim_{(\varepsilon,\nu)\to(0,0)} \Theta_{\varepsilon,\nu} = \frac{\pi}{2}$.

(ii) if $\lim_{(\varepsilon,\nu)\to(0,0)} \dfrac{\varepsilon^{2-\alpha}}{\nu^2} = \delta^2 \in \,]0,+\infty[$ *then* $\displaystyle\lim_{(\varepsilon,\nu)\to(0,0)} \Theta_{\varepsilon,\nu} =: \widetilde{\Theta}(\delta) \in \left]\frac{\pi}{2}, \frac{\pi}{2-\alpha}\right[$.
Moreover

$$\lim_{\delta\to 0} \widetilde{\Theta}(\delta) = \frac{\pi}{2-\alpha}, \qquad \lim_{\delta\to\infty} \widetilde{\Theta}(\delta) = \frac{\pi}{2}, \tag{11.53}$$

and $\widetilde{\Theta}(\delta)$ *is a strictly decreasing function of* δ*;*

(iii) if $\lim_{(\varepsilon,\nu)\to(0,0)} \dfrac{\varepsilon^{2-\alpha}}{\nu^2} = 0$ *then* $\displaystyle\lim_{(\varepsilon,\nu)\to(0,0)} \Theta_{\varepsilon,\nu} = \frac{\pi}{2-\alpha}$.

Proof. In the following we let χ_A be the characteristic function of the set $A \subseteq \mathbb{R}$. For each sequence $(\varepsilon_k, \nu_k) \to (0,0)$ we set $r_k := r_+(\varepsilon_k, \nu_k, \alpha)$. Define also the sequence (f_k) of functions by

$$f_k(\rho) := f_{\varepsilon_k,\nu_k}(\rho)\, \chi_{]1,1/r_k]}(\rho), \qquad \rho \in \,]1,+\infty[. \tag{11.54}$$

Let us prove assertion (i). By (11.33) and (11.34) we have

$$\Theta_{\varepsilon,\nu} = \frac{\nu\varepsilon^{\alpha/2}}{r_+} \int_1^{1/r_+} \frac{\left(1+(r_+^2/\varepsilon^2)\rho^2\right)^{\alpha/4}}{\rho\sqrt{(\rho^2-1)(1-r_+\rho)}} \frac{1}{\sqrt{K_{\varepsilon,\nu,\alpha}(r_+\rho)}}\, d\rho. \tag{11.55}$$

Let us now compute the pointwise limit, as $k \to \infty$, of the function sequence (f_k). Using (11.21), Proposition 13 and the relation

$$\lim_{k\to\infty} \frac{r_k^2}{\varepsilon_k^2} = \lim_{k\to\infty} \frac{\varepsilon_k^\alpha \nu_k^2}{\varepsilon_k^2} = 0. \tag{11.56}$$

we have

$$\lim_{k\to\infty} f_k(\rho) = \frac{1}{\rho\sqrt{\rho^2-1}}, \qquad \rho \in \,]1,+\infty[. \tag{11.57}$$

Since

$$\int_1^{+\infty} \frac{1}{\rho\sqrt{\rho^2-1}}\, d\rho = \frac{\pi}{2}, \tag{11.58}$$

to conclude the proof of (i) we have to show that (f_k) converges to $1/(\rho\sqrt{\rho^2-1})$ in $L^1(]1,+\infty[)$, i.e., we need to pass to the limit under the integral sign in (11.55). To this purpose, it is enough to prove that there exists a sequence (ξ_k) of functions defined on $]1,+\infty[$ such that $f_k \le \xi_k$ for any k and with (ξ_k) converging to a function ξ in $L^1(]1,+\infty[)$. Using (11.21), (11.56) and Proposition 12, we can say that there exist $\overline{a} > 0$ and two positive constant C_1, C_2 such that for each $(\varepsilon_k, \nu_k) \in]0, \overline{a}[^2$ (i.e., for k large enough) and for each $\rho \in [1, +\infty[$ we have

$$\begin{aligned} f_k(\rho) &= \frac{\nu_k \varepsilon_k^{\alpha/2}}{r_k} \frac{\left(1 + (r_k^2/\varepsilon_k^2)\rho^2\right)^{\alpha/4}}{\rho\sqrt{(\rho^2-1)(1-r_+\rho)}} \frac{1}{\sqrt{K_{\varepsilon_k,\nu_k,\alpha}(r_k\rho)}} \leq \\ &\leq \frac{C_1}{\rho\sqrt{\rho^2-1}} \chi_{[1,\varepsilon_k/r_k[}(\rho) \;+\; C_2 \frac{r_k^{\alpha/2}}{\varepsilon_k^{\alpha/2}} \frac{\rho^{\alpha/2-2}}{\sqrt{1-r_k\rho}} \chi_{[\varepsilon_k/r_k, 1/r_k[}(\rho) := \xi_k(\rho)', \end{aligned} \tag{11.59}$$

in fact we have

$$\left(1 + \frac{r_k^2}{\varepsilon_k^2}\right) \rho^{2^{\alpha/4}} \leq 2^{\alpha/4} \qquad\qquad \text{for } 1 \leq \rho \leq \varepsilon_k/r_k$$

and

$$\left(1 + \frac{r_k^2}{\varepsilon_k^2}\right) \rho^{2^{\alpha/4}} \frac{1}{\rho\sqrt{\rho^2-1}} \leq \tilde{C}_2 \rho^{\alpha/2-2} \qquad\qquad \text{for } \varepsilon_k/r_k \leq \rho \leq 1/r_k$$

and for some constant $\tilde{C}_2 > 0$.

We notice that

$$\lim_{k\to\infty} \xi_k(\rho) = \frac{C_1}{\rho\sqrt{\rho^2-1}} := \xi(\rho), \qquad \rho \in]1, +\infty[,$$

and clearly $\xi \in L^1(]1, +\infty[)$; in addition

$$\begin{aligned} \int_1^{+\infty} |\xi_k(\rho) - \xi(\rho)| \; d\rho \;\; &= \\ &= \int_{\varepsilon_k/r_k}^{1/r_k} \left| C_2 \frac{r_k^{\alpha/2}}{\varepsilon_k^{\alpha/2}} \frac{\rho^{\alpha/2-2}}{\sqrt{1-r_k\rho}} - C_1 \frac{1}{\rho\sqrt{\rho^2-1}} \right| d\rho + C_1 \int_{1/r_k}^{+\infty} \frac{1}{\rho\sqrt{\rho^2-1}} d\rho \leq \\ &\leq C_2 \frac{r_k^{\alpha/2}}{\varepsilon_k^{\alpha/2}} \int_{\varepsilon_k/r_k}^{1/r_k} \frac{\rho^{\alpha/2-2}}{\sqrt{1-r_k\rho}} d\rho + C_1 \int_{\varepsilon_k/r_k}^{+\infty} \frac{1}{\rho\sqrt{\rho^2-1}} d\rho. \end{aligned}$$

Hence

$$\lim_{k\to\infty} \int_1^{+\infty} |\xi_k(\rho) - \xi(\rho)| \; d\rho = 0$$

because of (11.56) and of

$$\int_{\varepsilon_k/r_k}^{1/r_k} \frac{1}{\rho^{2-\alpha/2}\sqrt{1-r_k\rho}} d\rho = \frac{1}{r_k^{\alpha/2-1}} \int_{\varepsilon_k}^{1} \frac{1}{r^{2-\alpha/2}\sqrt{1-r}} dr \leq C_3 \left(\frac{r_k}{\varepsilon_k}\right)^{(2-\alpha)/2} + C_4 r_k^{(2-\alpha)/2}, \tag{11.60}$$

for some positive constants C_3, C_4. This concludes the proof of (i).

To prove (ii) and (iii) we will use a similar strategy as in assertion (i), with however different computations.

Let us prove first assertion (iii). We write

$$\Theta_{\varepsilon,\nu} = \frac{\nu r_+^{\alpha/2}}{r_+}\int_1^{1/r_+} \frac{\left(\rho^2+(\varepsilon^2/r_+^2)\right)^{\alpha/4}}{\rho\sqrt{(\rho^2-1)(1-r_+\rho)}}\frac{1}{\sqrt{K_{\varepsilon,\nu,\alpha}(r_+\rho)}}\,d\rho, \tag{11.61}$$

and we define the sequence (f_k) as in (11.54) (notice that here r_k is infinitesimal with a different order). Using (11.24), Proposition 13 and the relation

$$\lim_{k\to\infty}\frac{\varepsilon_k^2}{r_k^2} = \lim_{k\to\infty}\left(\frac{\varepsilon_k^{2-\alpha}}{\nu_k^2}\right)^{2/(2-\alpha)} = 0. \tag{11.62}$$

we have

$$\lim_{k\to\infty} f_k(\rho) = \frac{\rho^{\alpha/2}}{\rho\sqrt{\rho^2-\rho^\alpha}}, \qquad \rho\in\,]1,+\infty[,$$

and

$$\int_1^{+\infty}\frac{\rho^{\alpha/2}}{\rho\sqrt{\rho^2-\rho^\alpha}}\,d\rho = \int_1^{+\infty}\frac{1}{\rho\sqrt{\rho^{2-\alpha}-1}}\,d\rho = \frac{\pi}{2-\alpha}.$$

Using (11.24), (11.62) and Proposition 12, we can say that there exist $\overline{a}>0$ and two positive constants C_1, C_2 such that for each $(\varepsilon_k,\nu_k)\in\,]0,\overline{a}[$ and for each $\rho\in[1,+\infty[$ we have

$$\begin{aligned} f_k(\rho) &= \frac{\nu_k r_k^{\alpha/2}}{r_k}\frac{\left(\rho^2+(\varepsilon_k^2/r_k^2)\right)^{\alpha/4}}{\rho\sqrt{(\rho^2-1)(1-r_k\rho)}}\frac{1}{\sqrt{K_{\varepsilon_k,\nu_k,\alpha}(r_k\rho)}} \le \\ &\le \frac{C_1\rho^{\alpha/2}}{\rho\sqrt{\rho^2-1}}\chi_{]1,1/\sqrt{r_k}[}(\rho) \;+\; \frac{C_2\rho^{\alpha/2-2}}{\sqrt{1-r_k\rho}}\chi_{]1/\sqrt{r_k},1/r_k[}(\rho) := \xi_k(\rho). \end{aligned}$$

We notice that

$$\lim_{k\to\infty}\xi_k(\rho) = \frac{C_1\rho^{\alpha/2}}{\rho\sqrt{\rho^2-1}} := \xi(\rho), \qquad \rho\in\,]1,+\infty[,$$

that $\xi\in L^1(]1,+\infty[)$ and that

$$\begin{aligned} \int_1^{+\infty}|\xi_k(\rho)-\xi(\rho)|\;d\rho &\le \int_{1/\sqrt{r_k}}^{1/r_k}\left|C_2\frac{\rho^{\alpha/2-2}}{\sqrt{1-r_k\rho}} - C_1\frac{\rho^{\alpha/2}}{\rho\sqrt{\rho^2-1}}\right|\,d\rho + C_1\int_{1/r_k}^{+\infty}\frac{\rho^{\alpha/2}}{\rho\sqrt{\rho^2-1}}\,d\rho \le \\ &\le C_2\int_{1/\sqrt{r_k}}^{1/r_k}\frac{\rho^{\alpha/2-2}}{\sqrt{1-r_k\rho}}\,d\rho + C_1\int_{1/\sqrt{r_k}}^{+\infty}\frac{\rho^{\alpha/2}}{\rho\sqrt{\rho^2-1}}\,d\rho. \end{aligned}$$

Therefore

$$\lim_{k\to\infty}\int_1^{+\infty} |\xi_k(\rho)-\xi(\rho)|\, d\rho = 0,$$

because

$$\int_{1/\sqrt{r_k}}^{1/r_k} \frac{1}{\rho^{2-\alpha/2}\sqrt{1-r_k\rho}}\, d\rho = r_k^{1-\alpha/2}\int_{\sqrt{r_k}}^{1}\frac{1}{r^{2-\alpha/2}\sqrt{1-r}}\, dr \le C_3 r_k^{1/2-\alpha/4}+C_4 r_k^{1-\alpha/2} = o(1), \tag{11.63}$$

for some positive constants C_3 and C_4. The proof of (iii) then follows as in (i).

Let us prove assertion (ii). We write $\Theta_{\varepsilon,\nu}$ as in (11.61), and we let (f_k) be as in (11.54). By (11.22) we have

$$\lim_{k\to\infty}\frac{\nu_k r_k^{\alpha/2}}{r_k} = \lim_{k\to\infty}\frac{\nu_k}{r_k^{1-\alpha/2}} = \frac{1}{\gamma^{(2-\alpha)/2}(\delta)}, \tag{11.64}$$

and

$$\lim_{k\to\infty}\frac{\varepsilon_k^2}{r_k^2} = \lim_{k\to\infty}\left[\frac{(\delta\nu_k)^{4/(2-\alpha)}}{\gamma^2(\delta)\nu_k^{4/(2-\alpha)}}\right] = \frac{\delta^{4/(2-\alpha)}}{\gamma^2(\delta)}. \tag{11.65}$$

Using relation (11.22), Proposition 13, (11.64) and (11.65) we have for any $\delta>0$,

$$\lim_{k\to\infty} f_k(\rho) = \sqrt{\Psi(\rho,\delta)}, \qquad \rho\in\,]1,+\infty[,$$

where we have set

$$\Psi(\rho,\delta) := \frac{\left[\rho^2\gamma^2(\delta)+\delta^{4/(2-\alpha)}\right]^{\alpha/2}}{\rho^2\left\{\rho^2\gamma^2(\delta)-\left[\rho^2\gamma^2(\delta)+\delta^{4/(2-\alpha)}\right]^{\alpha/2}\right\}}, \qquad \rho\in\,]1,+\infty[,\ \delta\in\,]0,+\infty[, \tag{11.66}$$

Define

$$\widetilde{\Theta}(\delta) := \int_1^{+\infty}\sqrt{\Psi(\rho,\delta)}\, d\rho, \qquad \delta\in\,]0,+\infty[. \tag{11.67}$$

We have

$$\begin{aligned} f_k(\rho) &= \frac{\nu r_k^{\alpha/2}}{r_k}\frac{\left(\rho^2+(\varepsilon_k^2/r_k^2)\right)^{\alpha/4}}{\rho\sqrt{(\rho^2-1)(1-r_k\rho)}}\frac{1}{\sqrt{K_{\varepsilon_k,\nu_k,\alpha}(r_k\rho)}} \le \\ &\le \frac{C_1\rho^{\alpha/2}}{\rho\sqrt{\rho^2-1}}\chi_{]1,1/\sqrt{r_k}[}(\rho) + \frac{C_2\rho^{\alpha/2-2}}{\sqrt{1-r_k\rho}}\chi_{[1/\sqrt{r_k},1/r_k[}(\rho), \end{aligned}$$

for some positive constants C_1, C_2. Then the assertion that $\Theta_{\varepsilon,\nu}$ converges to $\widetilde{\Theta}$ can be proved as in assertion (iii).

To conclude the proof of (ii) we have to prove that $\widetilde{\Theta} \in]\pi/2, \pi/(2-\alpha)[$, and that $\widetilde{\Theta}$ is strictly decreasing.

We observe that for any $\rho > 1$ the function $\delta \to \Psi(\rho,\delta)$ is strictly decreasing. Indeed using the relation

$$\delta^{4/(2-\alpha)} = \gamma^{4/\alpha}(\delta) - \gamma^2(\delta) = \gamma^2(\delta)\left[\gamma^{(4/\alpha)-2}(\delta) - 1\right]$$

we can write $\Psi(\rho,\delta)$ as a function of ρ and γ (still denoted by $\Psi(\rho,\gamma)$): we have, for $\rho > 1$,

$$\Psi(\rho,\gamma) = \frac{\left[\rho^2 - 1 + \gamma^{(4/\alpha)-2}\right]^{\alpha/2}}{\rho^2\left\{\rho^2\gamma^{2-\alpha} - \left[\rho^2 - 1 + \gamma^{(4/\alpha)-2}\right]^{\alpha/2}\right\}}, \tag{11.68}$$

and

$$\frac{\partial\Psi}{\partial\gamma}(\rho,\gamma) = \frac{\left(\rho^2 - 1 + \gamma^{(4/\alpha)-2}\right)^{\alpha/2-1}}{\left[\rho^2\gamma^{2-\alpha} - \left(\rho^2 - 1 + \gamma^{(4/\alpha)-2}\right)^{\alpha/2}\right]^2}\left\{-\left[\rho^2 - 1\right](2-\alpha)\gamma^{1-\alpha}\right\} < 0.$$

The monotonicity of Ψ then follows from monotonicity of γ; from this, the monotonicity of $\widetilde{\Theta}$ follows.

Observe now that, as

$$\lim_{\delta\to 0^+}\gamma(\delta) = 1 \qquad\qquad \lim_{\delta\to+\infty}\gamma(\delta) = +\infty$$

$$\lim_{\delta\to 0^+}\Psi(\rho,\gamma(\delta)) = \frac{\rho^\alpha}{\rho^2\left(\rho^2 - \rho^\alpha\right)}$$

and

$$\lim_{\delta\to+\infty}\Psi(\rho,\gamma(\delta)) = \lim_{\gamma\to+\infty}\Psi(\rho,\gamma) = \lim_{\gamma\to+\infty}\frac{\left[1 + \dfrac{(\rho^2-1)}{\gamma^{2(2-\alpha)/\alpha}}\right]^{\alpha/2}}{\rho^2\left[\rho^2 - \left[1 + \dfrac{(\rho^2-1)}{\gamma^{2(2-\alpha)/\alpha}}\right]^{\alpha/2}\right]} = \frac{1}{\rho^2\left(\rho^2-1\right)},$$

for each $\rho > 1$.

As $\lim_{\delta\to 0^+}\gamma(\delta) = 1$, we have

$$\rho^2 + \gamma^{2(2-\alpha)/\alpha} - 1 \le 2\rho^2$$
$$\rho^2\gamma^{2-\alpha} - \left[\rho^2 - 1 + \gamma^{2(2-\alpha)/\alpha}\right]^{\alpha/2} \ge \frac{\rho^2}{2}$$

for δ small enough and ρ large enough, so that

$$\sqrt{\Psi(\rho,\gamma)} \le \frac{C_1}{\rho^{2-(\alpha/2)}},$$

for a suitable constant $C_1 > 0$. We can then pass to the limit under the integral sign in (11.67), and we obtain

$$\lim_{\delta\to 0^+} \widetilde{\Theta}(\delta) = \int_1^{+\infty} \frac{\rho^{\alpha/2}}{\rho\sqrt{(\rho^2-\rho^\alpha)}}\,d\rho = \frac{\pi}{2-\alpha}.$$

On the other hand, as $\lim_{\delta\to+\infty}\gamma(\delta) = +\infty$, we have

$$\left[1+\frac{(\rho^2-1)}{\gamma^{2(2-\alpha)/\alpha}}\right] \leq 1+\rho^2-1 = \rho^2$$
$$\rho^2 - \left[1+\frac{(\rho^2-1)}{\gamma^{2(2-\alpha)/\alpha}}\right]^{\alpha/2} \geq \rho^2/2,$$

for δ and ρ large enough, so that

$$\sqrt{\Psi(\rho,\gamma)} \leq \frac{1}{\rho^{2-(\alpha/2)}}.$$

Then we can pass to the limit under the integral sign in (11.67), and we obtain

$$\lim_{\delta\to+\infty} \widetilde{\Theta}(\delta) = \int_1^{+\infty} \frac{1}{\rho\sqrt{\rho^2-1}}\,d\rho = \frac{\pi}{2}.$$

This concludes the proof of (iii) and of the theorem. □

We consider now the case $\alpha > 2$.

THEOREM 20. *Let* $\alpha > 2$. *Then* $\lim_{(\varepsilon,\nu)\to(0,0)}\Theta_{\varepsilon,\nu} = \dfrac{\pi}{2}$.

Proof. The proof follows exactly the same steps of the proof of assertion (i) in Theorem 19, writing $\Theta_{\varepsilon,\nu}$ as in (11.55) and considering (f_k) as (11.54): it is enough to observe that (11.20) holds and that $r_k/\varepsilon_k \to 0$ as $k\to\infty$. □

REMARK 35. Notice that this result is independent of the rate of convergence of (ε,ν) to $(0,0)$.

The case $\alpha = 2$ is, in some sense, a limit case, and requires a special proof. We cannot repeat all the arguments of the case $\alpha > 2$ because the expression at the right of the equality in (11.60) is not bounded anymore.

THEOREM 21. *Let* $\alpha = 2$. *The following assertions hold:*

(i) If $\lim_{(\varepsilon,\nu)\to(0,0)} \dfrac{1}{\nu|\log\varepsilon|} = +\infty$ *then* $\lim_{(\varepsilon,\nu)\to(0,0)}\Theta_{\varepsilon,\nu} = \dfrac{\pi}{2}$.

(ii) *if* $\lim_{(\varepsilon,\nu)\to(0,0)} \dfrac{1}{\nu|\log\varepsilon|} = \delta \in]0,+\infty[$ *then*

$$\lim_{(\varepsilon,\nu)\to(0,0)} \Theta_{\varepsilon,\nu} = \frac{\pi}{2} + \frac{1}{\delta} := \widetilde{\Theta}_2(\delta)\ .$$

(iii) *if* $\lim_{(\varepsilon,\nu)\to(0,0)} \dfrac{1}{\nu|\log\varepsilon|} = 0$ *then* $\lim_{(\varepsilon,\nu)\to(0,0)} \Theta_{\varepsilon,\nu} = +\infty$.

Proof. The function $K_{\varepsilon,\nu,\alpha}$ in (11.31) becomes

$$K_{\varepsilon,\nu,2}(r) = \frac{\big(1-\nu^2(1+\varepsilon^2)\big)}{(1+\varepsilon^2)}(1+r)\,,$$

and we can also explicitly compute

$$r_+ = r_+(\varepsilon,\nu,2) = \varepsilon\nu\frac{\sqrt{1+\varepsilon^2}}{\sqrt{1-\nu^2(1+\varepsilon^2)}}.$$

Therefore (11.33) holds with

$$f_{\varepsilon,\nu}(\rho) := \frac{\sqrt{1+(r_+^2/\varepsilon^2)\rho^2}}{\rho\sqrt{(\rho^2-1)(1-r_+^2\rho^2)}} \tag{11.69}$$

For each sequence $(\varepsilon_k,\nu_k)\to(0,0)$, we set $r_k := r_+(\varepsilon_k,\nu_k,2)$ and $f_k(\rho) := f_{\varepsilon_k,\nu_k}(\rho)\chi_{]1,1/r_k[}(\rho)$. We have

$$\lim_{k\to\infty} f_k(\rho) = \frac{1}{\rho\sqrt{\rho^2-1}}, \qquad \rho\in\,]1,+\infty[. \tag{11.70}$$

Let us prove assertion (i). For k large enough we have

$$f_k(\rho) \le \frac{C_1}{\rho\sqrt{\rho^2-1}}\chi_{]1,\varepsilon_k/r_k[}(\rho) + \frac{C_2\nu_k}{\rho\sqrt{1-r_k^2\rho^2}}\chi_{[\varepsilon_k/r_k,1/r_k[}(\rho) := \zeta_k(\rho)$$

for some positive constants C_1, C_2.

We notice that

$$\lim_{k\to\infty}\zeta_k(\rho) = \frac{C_1}{\rho\sqrt{\rho^2-1}} := \zeta(\rho), \qquad \rho\in\,]1,+\infty[.$$

and

$$\int_1^{+\infty} |\zeta_k(\rho)-\zeta(\rho)| \le C_2\nu_k\int_{\varepsilon_k/r_k}^{1/r_k}\frac{1}{\rho\sqrt{1-r_k^2\rho^2}}\,d\rho + C_1\int_{\varepsilon_k/r_k}^{+\infty}\frac{1}{\rho\sqrt{\rho^2-1}}\,d\rho. \tag{11.71}$$

Therefore

$$\lim_{k\to\infty}\int_1^{+\infty}|\zeta_k(\rho)-\zeta(\rho)|=0$$

because

$$\int_{\varepsilon_k/r_k}^{1/r_k}\frac{1}{\rho\sqrt{1-r_k^2\rho^2}}\,d\rho=\int_{\varepsilon_k}^{1}\frac{1}{r\sqrt{1-r^2}}\,dr=C_3|\log\varepsilon_k|+C_4, \tag{11.72}$$

$\lim_{k\to\infty}\varepsilon_k/r_k=0$ and $\lim_{k\to\infty}\nu_k|\log\varepsilon_k|=0$. Assertion (i) then follows passing to the limit under the integral sign in (11.33).

Let us prove assertion (ii). For each pair of constants C_1,C_2 satisfying

$$0<C_1<1<C_2, \tag{11.73}$$

we can find $\overline{a}>0$ such that for each $(\varepsilon_k,\nu_k)\in\,]0,\overline{a}[^2$ we have

$$\phi_k(\rho)\,\chi_{]1,\varepsilon_k/r_k[}(\rho)+C_1\psi_k(\rho)\,\chi_{[\varepsilon_k/r_k,1/r_k[}(\rho)\le f_k(\rho)\le\phi_k(\rho)\,\chi_{]1,\varepsilon_k/r_k[}(\rho)+C_2\psi_k(\rho)\,\chi_{[\varepsilon_k/r_k,1/r_k[}(\rho), \tag{11.74}$$

where

$$\phi_k(\rho):=\frac{\sqrt{1+(r_k^2/\varepsilon_k^2)\rho^2}}{\rho\sqrt{(\rho^2-1)(1-r_k^2\rho^2)}},\qquad \psi_k(\rho):=\frac{\nu_k}{\rho\sqrt{1-r_k^2\rho^2}}.$$

Integrating relation (11.74), passing to the limit as $k\to\infty$, and observing that there exists $C_3>\sqrt{2}$ such that, for k large enough,

$$\phi_k(\rho)\le\frac{C_3}{\rho\sqrt{\rho^2-1}},$$

we get

$$\frac{\pi}{2}+C_1\frac{1}{\delta}\le\liminf_{k\to\infty}\Theta_{\varepsilon_k,\nu_k}\le\limsup_{k\to\infty}\Theta_{\varepsilon_k,\nu_k}\le\frac{\pi}{2}+C_2\frac{1}{\delta}. \tag{11.75}$$

Then assertion (ii) follows by (11.74) and the arbitrariness of C_1,C_2.

Let us prove assertion (iii). We have

$$f_k(\rho)\ge\frac{C_1\nu_k}{\rho\sqrt{1-r_k^2\rho^2}}\chi_{[\varepsilon_k/r_k,1/r_k[}(\rho),$$

for k large enough and for some constant $C_1>0$. Therefore

$$\lim_{k\to\infty}\Theta_{\varepsilon_k,\nu_k}\ge C_1\liminf_{k\to\infty}\nu_k\int_{\varepsilon_k/r_k}^{1/r_k}\frac{1}{\rho\sqrt{1-r_k^2\rho^2}}\,d\rho=\liminf_{k\to\infty}O(\nu_k|\log\varepsilon_k|)=+\infty.$$

□

THEOREM 22. *Let* $\alpha=2$. *If* $\nu\neq0$ *is constant then* $\lim_{\varepsilon\to0}\Theta_{\varepsilon,\nu}=+\infty$.

Proof. We write $\Theta_{\varepsilon,\nu}$ as in (11.33), with $f_{\varepsilon,\nu}(\rho)$ is the same function as in (11.69). For each sequence $\varepsilon_k \to 0$, we set $\tilde{r}_k := r_+(\varepsilon_k, \nu, 2)$ and we compute the limit, as $k \to \infty$, of the function sequence $\tilde{f}_k(\rho) := f_{\varepsilon_k,\nu}(\rho)\, \chi_{]1,1/r_k]}(\rho)$, defined for $\rho \in]1,+\infty[$. We have

$$\lim_{k\to\infty} \tilde{f}_k(\rho) = \frac{\sqrt{1+\nu^2\rho^2}}{\rho\sqrt{\rho^2-(1+\nu^2\rho^2)}} := f(\rho), \qquad \rho \in]1,+\infty[.$$

Since $f \notin L^1(]1,+\infty[)$, by Fatou's lemma we get

$$\liminf_{k\to\infty} \int_1^\infty f_k(\rho)\, d\rho \geq \int_1^\infty f(\rho)\, d\rho = +\infty.$$

□

REMARK 36. Notice that for $\varepsilon = 0$ we obtain the Cotes's spiral (see [82]).

REMARK 37. We do some considerations related to the results just described.

First of all we notice that $\alpha = 2$ is a limit case between two very different kinds of behavior, and this happens also for the non-regularized problem (11.1).

Furthermore, following [45], in Chapter 2 we have described the main two kinds of regularization:

- *branch regularization (Sundman)*: it is possible only when

$$\alpha = 2\left[\left(\frac{q}{p}\right) - 1\right]$$

  where p, q are relatively prime integers with q odd. If p is even then we obtain the reflection solution, if p is odd then we obtain the transmission solution;

- *block regularization (Easton)*: it is possible only when

$$\alpha = 2\left(1 - \frac{1}{n}\right)$$

  where n is a positive integer.

We notice that the last relation corresponds to $n = 2/(2-\alpha)$, that curiously represents the delimiting exponent of ε with respect to ν in the three cases of Proposition (11), and that the angle of maximal deflection of the limit trajectory in the case $0 < \alpha < 2$ is given by the increment $2\lim_{\varepsilon,\nu} \Theta_{\varepsilon,\nu} = 2\pi/(2-\alpha)$. This means that Easton's regularization is possible only when the limit trajectory has a maximal deflection of a multiple integer of π (transmission or reflection solution) when crossing the origin of the reference system.

Appendix A

In this chapter we prove that the average of the indirect part of the perturbation is zero in the general framework, with the planets on elliptic inclined orbits. Without loss of generality we can prove the result for only one perturbing planet.

A.1 Average of the indirect perturbation

The following theorem holds:

Theorem 23. The average of the indirect term of the perturbing function R (see (3.21)) over the angles ℓ, ℓ' is zero.

Proof. The indirect perturbation in a 3-body problem is the gravitational effect on one of the bodies due to the fact that the other two bodies interact, so that they do not attract the body from fixed positions in the configuration space.

If we consider a Sun-planet-asteroid problem in a heliocentric reference frame, the average of the indirect perturbation on the asteroid is given by

$$\overline{R_1} = -\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} k^2\mu \frac{< x', x >}{|x'|^3}\, d\ell d\ell' \tag{A.1}$$

where $x = (x_1, x_2, x_3)$ and $x' = (x'_1, x'_2, x'_3)$ are the positions of the asteroid and of the planet respectively.

In the mutual reference frame we can describe the Keplerian ellipses of the asteroid and of the planet by formulas (9.7) and (9.8), given in Chapter 9. Then we have

$$|x'|^3 = a'^3[1 - e'\cos u']^3$$

and

$$\begin{aligned} < x', x >= aa' \Big\{ & [(\cos u' - e')\cos\omega'_M - (1-e'^2)^{\frac{1}{2}}\sin u' \sin\omega'_M]\cdot \\ & \cdot[(\cos u - e)\cos\omega_M - (1-e^2)^{\frac{1}{2}}\sin u \sin\omega_M] + [(\cos u' - e')\sin\omega'_M + \\ & + (1-e'^2)^{\frac{1}{2}}\sin u' \cos\omega'_M]\cdot[(\cos u - e)\sin\omega_M + (1-e^2)^{\frac{1}{2}}\sin u \cos\omega_M]\cos I_M \Big\} . \end{aligned}$$

We pass to the variables $\ell = u - e\sin u$, $\ell' = u' - e'\sin u'$, defined by Kepler's equation, and we obtain

$$-\frac{a'^3(2\pi)^2}{k^2\mu}\overline{R_1} = \int_0^{2\pi}\int_0^{2\pi} < x', x > \frac{(1-e\cos u)}{(1-e'\cos u')^2}\,du\,du' =$$

$$= \int_0^{2\pi} f_1(u)\,du \int_0^{2\pi}\frac{\sin u'}{(1-e'\cos u')^2}\,du' + \int_0^{2\pi} f_2(u)\,du\int_0^{2\pi}\frac{(\cos u'-e')}{(1-e'\cos u')^2}\,du'$$

with f_1, f_2 integrable functions over the interval $[0, 2\pi]$.

We can easily prove the following formulas:

$$\mathfrak{I}_1 = \int_0^{2\pi}\frac{\sin u'}{(1-e'\cos u')^2}\,du' = 0 \qquad\qquad \mathfrak{I}_2 = \int_0^{2\pi}\frac{(\cos u'-e')}{(1-e'\cos u')^2}\,du' = 0.$$

The first result is immediate, because it is the integral of a periodic, odd function on its period. To compute the second integral we use the invariance under translations of the integration domain to do the integration over $[-\pi, \pi]$; then, by the change of variable $z = \tan(u'/2)$, we obtain

$$\mathfrak{I}_2 = 2\int_{-\infty}^{+\infty}\frac{K^2 - z^2}{[K^2+z^2]^2}\,dz$$

where $K = \sqrt{(1-e')/(1+e')}$ and, if we set $w = z/K$, we have

$$\mathfrak{I}_2 = \frac{4}{K}\int_{-\infty}^{+\infty}\frac{1}{[1+w^2]^2}\,dw - \frac{2\pi}{K}\,.$$

We notice that

$$\int_{-\infty}^{+\infty}\frac{1}{[1+w^2]^2}\,dw = \int_{-\infty}^{+\infty}\frac{1}{1+w^2}\,dw - \int_{-\infty}^{+\infty}\frac{w^2}{[1+w^2]^2}\,dw =$$

$$= \pi + \frac{1}{2}\int_{-\infty}^{+\infty} w\,d\Big[\frac{1}{1+w^2}\Big] = \pi + \frac{1}{2}\Big\{\lim_{w\to\infty}\frac{2w}{1+w^2} - \int_{-\infty}^{+\infty}\frac{1}{1+w^2}\,dw\Big\} = \frac{\pi}{2}\,.$$

It follows that $\mathfrak{I}_2 = 0$ and the proof is complete. □

Appendix B

B.1 Circular coplanar case

In this section we give the explicit formulas for the derivatives of the direct term of the perturbing function and for the approximations of this function at the ascending and descending node defined by the Wetherill functions. The framework is the one of the circular coplanar case.

B.1.1 Derivatives of $1/\mathfrak{D}$

We recall that the square of the distance $\mathfrak{D}$ is defined by

$$\begin{aligned}\mathfrak{D}^2(u,u') &= a^2(1-e\cos u)^2 + a'^2 - 2aa'\{\cos u'[(\cos u - e)\cos\omega - \beta\sin u\sin\omega] + \\ &\quad + \sin u'\cos I[(\cos u - e)\sin\omega + \beta\sin u\cos\omega]\}\end{aligned}$$

where $\beta = \sqrt{1-e^2}$.

DERIVATIVE WITH RESPECT TO e

$$\frac{\partial}{\partial e}\left(\frac{1}{\mathfrak{D}}\right) = -\frac{1}{2\mathfrak{D}^3}\frac{\partial}{\partial e}\mathfrak{D}^2$$

$$\begin{aligned}\frac{\partial}{\partial e}\mathfrak{D}^2 &= -2a^2(\cos u - e) - 2ea^2\sin^2 u + 2aa'\left\{\cos u'\cos\omega - \frac{e\sin\omega}{\beta}\sin u\cos u' + \right. \\ &\quad \left. + \sin\omega\cos I\sin u' + \frac{e\cos\omega}{\beta}\sin u\sin u'\cos I\right\}\end{aligned}$$

DERIVATIVE WITH RESPECT TO ω

$$\frac{\partial}{\partial \omega}\left(\frac{1}{\mathfrak{D}}\right) = -\frac{1}{2\mathfrak{D}^3}\frac{\partial}{\partial \omega}\mathfrak{D}^2$$

$$\begin{aligned}\frac{\partial}{\partial \omega}\mathfrak{D}^2 &= -2aa'\left\{\cos u'[(\cos u - e)(-\sin\omega) - \beta\sin u\cos\omega] + \right. \\ &\quad \left. + \sin u'\cos I[(\cos u - e)\cos\omega - \beta\sin u\sin\omega]\right\}\end{aligned}$$

DERIVATIVE WITH RESPECT TO I

$$\frac{\partial}{\partial I}\left(\frac{1}{\mathfrak{D}}\right) = -\frac{1}{2\mathfrak{D}^3}\frac{\partial}{\partial I}\mathfrak{D}^2$$
$$\frac{\partial}{\partial I}\mathfrak{D}^2 = 2aa'\sin u'\sin I\{(\cos u - e)\sin\omega + \beta\sin u\cos\omega\}$$

B.1.2 Derivatives of $1/\mathfrak{d}$

We recall that the square of the Wetherill function, for the approximation of the distance at the ascending node, is given by

$$\mathfrak{d}^2(u,u') = a'^2k'^2 + [\mathcal{F}^2 + \mathcal{G}^2]k^2 - 2kk'[\mathcal{G}a'\cos I] - 2d^+_{nod}\mathcal{F}k + (d^+_{nod})^2$$

where

$$\begin{cases} k = \ell - \overline{\ell} = u - \overline{u} - e(\sin u - \sin\overline{u}) \\ k' = \ell' - \overline{\ell}' = u' - \overline{u}' - e'(\sin u' - \sin\overline{u}') \end{cases}$$

DERIVATIVE WITH RESPECT TO e

$$\frac{\partial}{\partial e}\left(\frac{1}{\mathfrak{d}}\right) = -\frac{1}{2\mathfrak{d}^3}\frac{\partial}{\partial e}\mathfrak{d}^2$$

$$\begin{aligned}\frac{\partial}{\partial e}\mathfrak{d}^2 \quad = \quad & 2a'^2k'\frac{\partial k'}{\partial e} + 2k\frac{\partial k}{\partial e}[\mathcal{F}^2 + \mathcal{G}^2] + k^2\frac{\partial}{\partial e}\left([\mathcal{F}^2 + \mathcal{G}^2]\right) - \\ & -2\frac{\partial k}{\partial e}k'[\mathcal{G}a'\cos I] - 2k\frac{\partial k'}{\partial e}[\mathcal{G}a'\cos I] - 2kk'\frac{\partial}{\partial e}\left([\mathcal{G}a'\cos I]\right) - \\ & -2\frac{\partial}{\partial e}\left(d^+_{nod}\right)\mathcal{F}k - 2d^+_{nod}\frac{\partial\mathcal{F}}{\partial e}k - 2d^+_{nod}\mathcal{F}\frac{\partial k}{\partial e} + 2d^+_{nod}\frac{\partial}{\partial e}\left(d^+_{nod}\right)\end{aligned}$$

in which

$$\frac{\partial k}{\partial e} = -(1 - e\cos\overline{u})\frac{\partial\overline{u}}{\partial e} - (\sin u - \sin\overline{u}); \qquad \frac{\partial\overline{u}}{\partial e} = \frac{\sin\omega}{\beta(1 + e\cos\omega)}; \qquad \frac{\partial k'}{\partial e} = 0;$$
$$\frac{\partial}{\partial e}\left([\mathcal{F}^2 + \mathcal{G}^2]\right) = 2\left[\mathcal{F}\frac{\partial\mathcal{F}}{\partial e} + \mathcal{G}\frac{\partial\mathcal{G}}{\partial e}\right]; \qquad \frac{\partial}{\partial e}\left([\mathcal{G}a'\cos I]\right) = a'\cos I\frac{\partial\mathcal{G}}{\partial e};$$
$$\frac{\partial\mathcal{F}}{\partial e} = \frac{a\sin\omega}{\beta^3}; \qquad \frac{\partial\mathcal{G}}{\partial e} = \frac{a(\cos\omega + e)}{\beta^3};$$
$$\frac{\partial}{\partial e}\left(d^+_{nod}\right) = -\frac{a[\cos\omega(1 + e^2) + 2e]}{(1 + e\cos\omega)^2}.$$

DERIVATIVE WITH RESPECT TO ω

$$\frac{\partial}{\partial\omega}\left(\frac{1}{\mathfrak{d}}\right) = -\frac{1}{2\mathfrak{d}^3}\frac{\partial}{\partial\omega}\mathfrak{d}^2$$

$$\begin{aligned}\frac{\partial}{\partial\omega}\mathfrak{d}^2 = 2a'^2k'\frac{\partial k'}{\partial\omega} + 2k\frac{\partial k}{\partial\omega}[\mathcal{F}^2+\mathcal{G}^2] + k^2\frac{\partial}{\partial\omega}\left([\mathcal{F}^2+\mathcal{G}^2]\right) - \\ -2\frac{\partial k}{\partial\omega}k'[\mathcal{G}a'\cos I] - 2k\frac{\partial k'}{\partial\omega}[\mathcal{G}a'\cos I] - 2kk'\frac{\partial}{\partial\omega}\left([\mathcal{G}a'\cos I]\right) - \\ -2\frac{\partial}{\partial\omega}\left(d^+_{nod}\right)\mathcal{F}k - 2d^+_{nod}\frac{\partial\mathcal{F}}{\partial\omega}k - 2d^+_{nod}\mathcal{F}\frac{\partial k}{\partial\omega} + 2d^+_{nod}\frac{\partial}{\partial\omega}\left(d^+_{nod}\right)\end{aligned}$$

in which

$$\begin{aligned}&\frac{\partial k}{\partial\omega} = -(1-e\cos\overline{u})\frac{\partial\overline{u}}{\partial\omega}; \qquad \frac{\partial\overline{u}}{\partial\omega} = -\frac{\beta}{1+e\cos\omega}; \qquad \frac{\partial k'}{\partial\omega} = 0;\\ &\frac{\partial}{\partial\omega}\left([\mathcal{F}^2+\mathcal{G}^2]\right) = 2\left[\mathcal{F}\frac{\partial\mathcal{F}}{\partial\omega} + \mathcal{G}\frac{\partial\mathcal{G}}{\partial\omega}\right]; \qquad \frac{\partial}{\partial\omega}\left([\mathcal{G}a'\cos I]\right) = a'\cos I\frac{\partial\mathcal{G}}{\partial\omega};\\ &\frac{\partial\mathcal{F}}{\partial\omega} = \frac{ae\cos\omega}{\beta}; \qquad \frac{\partial\mathcal{G}}{\partial\omega} = -\frac{ae\sin\omega}{\beta};\\ &\frac{\partial}{\partial\omega}\left(d^+_{nod}\right) = \frac{ae\beta^2\sin\omega}{(1+e\cos\omega)^2}\ .\end{aligned}$$

B.1.3 Derivatives of $1/\mathfrak{d}_1$

We recall that the square of the Wetherill function, for the approximation of the distance at the descending node, is given by

$$\mathfrak{d}_1^2(u,u') = a'^2k_1'^2 + [\mathcal{F}_1^2+\mathcal{G}_1^2]k_1^2 + 2k_1k_1'[\mathcal{G}_1a'\cos I] + 2d^-_{nod}\mathcal{F}_1k_1 + (d^-_{nod})^2$$

where

$$\begin{cases} k_1 = \ell - \overline{\ell}_1 = u - \overline{u}_1 - e(\sin u - \sin\overline{u}_1)\\ k_1' = \ell' - \overline{\ell}_1' = u' - \overline{u}_1' - e'(\sin u' - \sin\overline{u}_1')\end{cases}$$

DERIVATIVE WITH RESPECT TO e

$$\frac{\partial}{\partial e}\left(\frac{1}{\mathfrak{d}_1}\right) = -\frac{1}{2\mathfrak{d}_1^3}\frac{\partial}{\partial e}\mathfrak{d}_1^2$$

$$\begin{aligned}\frac{\partial}{\partial e}\mathfrak{d}_1^2 \;=\; & 2a'^2k_1'\frac{\partial k_1'}{\partial e} + 2k_1\frac{\partial k_1}{\partial e}[\mathcal{F}_1^2+\mathcal{G}_1^2] + k_1^2\frac{\partial}{\partial e}\left([\mathcal{F}_1^2+\mathcal{G}_1^2]\right) + \\ & +2\frac{\partial k_1}{\partial e}k_1'[\mathcal{G}_1a'\cos I] + 2k_1\frac{\partial k_1'}{\partial e}[\mathcal{G}_1a'\cos I] + 2k_1k_1'\frac{\partial}{\partial e}\left([\mathcal{G}_1a'\cos I]\right) - \\ & +2\frac{\partial}{\partial e}\left(d^-_{nod}\right)\mathcal{F}_1k_1 + 2d^-_{nod}\frac{\partial\mathcal{F}_1}{\partial e}k_1 + 2d^-_{nod}\mathcal{F}_1\frac{\partial k_1}{\partial e} + 2d^-_{nod}\frac{\partial}{\partial e}\left(d^-_{nod}\right)\end{aligned}$$

in which

$$\begin{aligned}&\frac{\partial k_1}{\partial e} = -(1-e\cos\overline{u}_1)\frac{\partial\overline{u}_1}{\partial e} - (\sin u - \sin\overline{u}_1); \qquad \frac{\partial\overline{u}_1}{\partial e} = -\frac{\sin\omega}{\beta(1-e\cos\omega)}; \qquad \frac{\partial k_1'}{\partial e} = 0;\\ &\frac{\partial}{\partial e}\left([\mathcal{F}_1^2+\mathcal{G}_1^2]\right) = 2\left[\mathcal{F}_1\frac{\partial\mathcal{F}_1}{\partial e} + \mathcal{G}_1\frac{\partial\mathcal{G}_1}{\partial e}\right]; \qquad \frac{\partial}{\partial e}\left([\mathcal{G}_1a'\cos I]\right) = a'\cos I\frac{\partial\mathcal{G}_1}{\partial e};\\ &\frac{\partial\mathcal{F}_1}{\partial e} = \frac{a\sin\omega}{\beta^3}; \qquad \frac{\partial\mathcal{G}_1}{\partial e} = \frac{a(\cos\omega - e)}{\beta^3};\\ &\frac{\partial}{\partial e}\left(d^-_{nod}\right) = \frac{a[\cos\omega(1+e^2)-2e]}{(1-e\cos\omega)^2}\ .\end{aligned}$$

DERIVATIVE WITH RESPECT TO ω

$$
\begin{aligned}
&\frac{\partial}{\partial\omega}\left(\frac{1}{\eth_1}\right) = -\frac{1}{2\eth_1^3}\frac{\partial}{\partial\omega}\eth_1^2 \\
&\frac{\partial}{\partial\omega}\eth_1^2 = 2a'^2k_1'\frac{\partial k_1'}{\partial\omega} + 2k_1\frac{\partial k_1}{\partial\omega}[\mathcal{F}_1^2+\mathcal{G}_1^2] + k_1^2\frac{\partial}{\partial\omega}\left([\mathcal{F}_1^2+\mathcal{G}_1^2]\right) + \\
&+2\frac{\partial k_1}{\partial\omega}k_1'[\mathcal{G}_1 a'\cos I] + 2k_1\frac{\partial k_1'}{\partial\omega}[\mathcal{G}_1 a'\cos I] + 2k_1k_1'\frac{\partial}{\partial\omega}\left([\mathcal{G}_1 a'\cos I]\right) + \\
&+2\frac{\partial}{\partial\omega}\left(d^-_{nod}\right)\mathcal{F}_1 k_1 + 2d^-_{nod}\frac{\partial\mathcal{F}_1}{\partial\omega}k_1 + 2d^-_{nod}\mathcal{F}_1\frac{\partial k_1}{\partial\omega} + 2d^-_{nod}\frac{\partial}{\partial\omega}\left(d^-_{nod}\right)
\end{aligned}
$$

in which

$$
\begin{aligned}
&\frac{\partial k_1}{\partial\omega} = -(1-e\cos\overline{u}_1)\frac{\partial\overline{u}_1}{\partial\omega}; \qquad \frac{\partial\overline{u}_1}{\partial\omega} = -\frac{\beta}{1-e\cos\omega}; \qquad \frac{\partial k_1'}{\partial\omega} = 0; \\
&\frac{\partial}{\partial\omega}\left([\mathcal{F}_1^2+\mathcal{G}_1^2]\right) = 2\left[\mathcal{F}_1\frac{\partial\mathcal{F}_1}{\partial\omega} + \mathcal{G}_1\frac{\partial\mathcal{G}_1}{\partial\omega}\right]; \qquad \frac{\partial}{\partial\omega}\left([\mathcal{G}_1 a'\cos I]\right) = a'\cos I\frac{\partial\mathcal{G}_1}{\partial\omega}; \\
&\frac{\partial\mathcal{F}_1}{\partial\omega} = \frac{ae\cos\omega}{\beta}; \qquad \frac{\partial\mathcal{G}_1}{\partial\omega} = -\frac{ae\sin\omega}{\beta}; \\
&\frac{\partial}{\partial\omega}\left(d^-_{nod}\right) = -\frac{ae\beta^2\sin\omega}{(1-e\cos\omega)^2}.
\end{aligned}
$$

B.1.4 Derivatives of $\int(1/\mathrm{d})\,d\ell d\ell'$

We write down the formulas needed for the computation of the derivatives of $\int_{\mathbb{T}^2}\frac{1}{\mathrm{d}}\,d\ell d\ell'$. The contents of this section has been used to write the FORTRAN programs to compute the weak averaged solutions.

DERIVATIVE WITH RESPECT TO e

$$
\begin{aligned}
\frac{\partial}{\partial e}\int_{\mathbb{T}^2}\frac{1}{\mathrm{d}}\,d\ell d\ell' \;=\; & \frac{\partial}{\partial e}\left[\frac{1}{\sqrt{det(\mathcal{A})}}\right]\cdot\left\{\sum_{j=1}^{4}\int_{\theta_j}^{\theta_{j+1}}\sqrt{\left(d^+_{min}\right)^2 + r_j^2(\theta)}\,d\theta - 2\pi d^+_{min}\right\} + \\
& + \left[\frac{1}{\sqrt{det(\mathcal{A})}}\right]\cdot\left\{\frac{1}{2}\sum_{j=1}^{4}\int_{\theta_j}^{\theta_{j+1}}\frac{\frac{\partial}{\partial e}[(d^+_{min})^2 + r_j^2(\theta)]}{\sqrt{\left(d^+_{min}\right)^2 + r_j^2(\theta)}}\,d\theta - 2\pi\frac{\partial}{\partial e}d^+_{min}\right\}.
\end{aligned}
$$

The derivatives appearing in the previous formula are the following:

$$
\begin{aligned}
&\frac{\partial}{\partial e}\left[\frac{1}{\sqrt{det(\mathcal{A})}}\right] = -\frac{1}{2[det(\mathcal{A})]^{3/2}}\frac{\partial}{\partial e}\left(det(\mathcal{A})\right); \\
&\frac{\partial}{\partial e}\left[\left(d^+_{min}\right)^2\right] = 2d^+_{nod}\frac{\partial}{\partial e}\left(d^+_{nod}\right)\left[\frac{a'^2\mathcal{G}^2\sin^2 I}{det(\mathcal{A})}\right] + \left(d^+_{nod}\right)^2\frac{a'^2\sin^2 I}{det(\mathcal{A})^2}\left[2det(\mathcal{A})\mathcal{G}\frac{\partial\mathcal{G}}{\partial e} - \mathcal{G}^2\frac{\partial}{\partial e}\left(det(\mathcal{A})\right)\right]; \\
&\frac{\partial}{\partial e}(det(\mathcal{A})) = 2a'^2\mathcal{F}\frac{\partial\mathcal{F}}{\partial e} + 2a'^2\mathcal{G}\frac{\partial\mathcal{G}}{\partial e}\sin^2 I;
\end{aligned}
$$

$$\frac{\partial S_1}{\partial e} = \frac{a'\cos I}{(det(\mathcal{A}))^2}\left\{det(\mathcal{A})\left[\frac{\partial \mathcal{F}}{\partial e}\mathcal{G}d^+_{nod} + \mathcal{F}\frac{\partial \mathcal{G}}{\partial e}d^+_{nod} + \mathcal{F}\,\mathcal{G}\frac{\partial}{\partial e}\left(d^+_{nod}\right)\right] - \mathcal{F}\,\mathcal{G}d^+_{nod}\frac{\partial}{\partial e}\left(det(\mathcal{A})\right)\right\};$$

$$\frac{\partial S_2}{\partial e} = \frac{a'^2}{(det(\mathcal{A}))^2}\left\{det(\mathcal{A})\left[\frac{\partial \mathcal{F}}{\partial e}d^+_{nod} + \mathcal{F}\frac{\partial}{\partial e}\left(d^+_{nod}\right)\right] - \mathcal{F}d^+_{nod}\frac{\partial}{\partial e}\left(det(\mathcal{A})\right)\right\};$$

$$\frac{\partial r_1}{\partial e} = \frac{\tau}{(\rho\cos\theta-\sigma\sin\theta)^2}\left\{(\rho\cos\theta-\sigma\sin\theta)\left[\frac{\partial\rho}{\partial e}(\pi-S_1)-\rho\frac{\partial S_1}{\partial e}\right]-\rho(\pi-S_1)\left[\frac{\partial\rho}{\partial e}\cos\theta-\frac{\partial\sigma}{\partial e}\sin\theta\right]\right\};$$

$$\frac{\partial r_2}{\partial e} = \frac{1}{\sin\theta}\left\{\frac{\partial\rho}{\partial e}(\pi-\overline{\ell}-S_2)-\rho\left(\frac{\partial\overline{\ell}}{\partial e}+\frac{\partial S_2}{\partial e}\right)\right\};$$

$$\frac{\partial r_3}{\partial e} = -\frac{\tau}{(\rho\cos\theta-\sigma\sin\theta)^2}\left\{(\rho\cos\theta-\sigma\sin\theta)\left[\frac{\partial\rho}{\partial e}(\pi+S_1)+\rho\frac{\partial S_1}{\partial e}\right]-\rho(\pi+S_1)\left[\frac{\partial\rho}{\partial e}\cos\theta-\frac{\partial\sigma}{\partial e}\sin\theta\right]\right\};$$

$$\frac{\partial r_4}{\partial e} = -\frac{1}{\sin\theta}\left\{\frac{\partial\rho}{\partial e}(\pi+\overline{\ell}+S_2)+\rho\left(\frac{\partial\overline{\ell}}{\partial e}+\frac{\partial S_2}{\partial e}\right)\right\};$$

$$\frac{\partial\tau}{\partial e}=0;\qquad \frac{\partial\sigma}{\partial e}=-\frac{a\cos I}{\beta^3}(\cos\omega+e);\qquad \frac{\partial\rho}{\partial e}=\frac{1}{2a'[det(\mathcal{A})]^{\frac{1}{2}}}\frac{\partial}{\partial e}\left(det(\mathcal{A})\right);$$

$$\frac{\partial\overline{\ell}}{\partial e}=-\frac{\partial k}{\partial e};\qquad \frac{\partial\overline{\ell}'}{\partial e}=0\,.$$

DERIVATIVE WITH RESPECT TO ω

$$\begin{aligned}\frac{\partial}{\partial\omega}\int_{\mathbb{T}^2}\frac{1}{\mathrm{d}}\,d\ell\,d\ell' \;=\;& \frac{\partial}{\partial\omega}\left[\frac{1}{\sqrt{det(\mathcal{A})}}\right]\cdot\left\{\sum_{j=1}^{4}\int_{\theta_j}^{\theta_{j+1}}\sqrt{\left(d^+_{min}\right)^2+r_j^2(\theta)}\,d\theta-2\pi d^+_{min}\right\}+\\ &+\left[\frac{1}{\sqrt{det(\mathcal{A})}}\right]\cdot\left\{\frac{1}{2}\sum_{j=1}^{4}\int_{\theta_j}^{\theta_{j+1}}\frac{\frac{\partial}{\partial\omega}[\left(d^+_{min}\right)^2+r_j^2(\theta)]}{\sqrt{\left(d^+_{min}\right)^2+r_j^2(\theta)}}\,d\theta-2\pi\frac{\partial}{\partial\omega}d^+_{min}\right\}\,.\end{aligned}$$

The derivatives appearing in the previous formula are the following:

$$\frac{\partial}{\partial\omega}\left[\frac{1}{\sqrt{det(\mathcal{A})}}\right]=-\frac{1}{2[det(\mathcal{A})]^{3/2}}\frac{\partial}{\partial\omega}\left(det(\mathcal{A})\right);$$

$$\frac{\partial}{\partial\omega}\left[\left(d^+_{min}\right)^2\right]=2d^+_{nod}\frac{\partial}{\partial\omega}\left(d^+_{nod}\right)\left[\frac{a'^2\mathcal{G}^2\sin^2 I}{det(\mathcal{A})}\right]+\left(d^+_{nod}\right)^2\frac{a'^2\sin^2 I}{det(\mathcal{A})^2}\left[2det(\mathcal{A})\,\mathcal{G}\frac{\partial\mathcal{G}}{\partial\omega}-\mathcal{G}^2\frac{\partial}{\partial\omega}\left(det(\mathcal{A})\right)\right];$$

$$\frac{\partial}{\partial\omega}(det(\mathcal{A}))=2a'^2\mathcal{F}\frac{\partial\mathcal{F}}{\partial\omega}+2a'^2\mathcal{G}\frac{\partial\mathcal{G}}{\partial\omega}\sin^2 I;$$

$$\frac{\partial S_1}{\partial\omega} = \frac{a'\cos I}{(det(\mathcal{A}))^2}\left\{det(\mathcal{A})\left[\frac{\partial \mathcal{F}}{\partial\omega}\mathcal{G}d^+_{nod} + \mathcal{F}\frac{\partial \mathcal{G}}{\partial\omega}d^+_{nod} + \mathcal{F}\,\mathcal{G}\frac{\partial}{\partial\omega}\left(d^+_{nod}\right)\right] - \mathcal{F}\,\mathcal{G}d^+_{nod}\frac{\partial}{\partial\omega}\left(det(\mathcal{A})\right)\right\};$$

$$\frac{\partial S_2}{\partial\omega} = \frac{a'^2}{(det(\mathcal{A}))^2}\left\{det(\mathcal{A})\left[\frac{\partial\mathcal{F}}{\partial\omega}d^+_{nod} + \mathcal{F}\frac{\partial}{\partial\omega}\left(d^+_{nod}\right)\right] - \mathcal{F}d^+_{nod}\frac{\partial}{\partial\omega}\left(det(\mathcal{A})\right)\right\};$$

$$\frac{\partial r_1}{\partial\omega} = \frac{\tau}{(\rho\cos\theta-\sigma\sin\theta)^2}\left\{(\rho\cos\theta-\sigma\sin\theta)\left[\frac{\partial\rho}{\partial\omega}(\pi-S_1)-\rho\frac{\partial S_1}{\partial\omega}\right]-\rho(\pi-S_1)\left[\frac{\partial\rho}{\partial\omega}\cos\theta-\frac{\partial\sigma}{\partial\omega}\sin\theta\right]\right\};$$

$$\frac{\partial r_2}{\partial\omega} = \frac{1}{\sin\theta}\left\{\frac{\partial\rho}{\partial\omega}(\pi-\bar{\ell}-S_2)-\rho\left(\frac{\partial\bar{\ell}}{\partial\omega}+\frac{\partial S_2}{\partial\omega}\right)\right\};$$

$$\frac{\partial r_3}{\partial\omega} = -\frac{\tau}{(\rho\cos\theta-\sigma\sin\theta)^2}\left\{(\rho\cos\theta-\sigma\sin\theta)\left[\frac{\partial\rho}{\partial\omega}(\pi+S_1)+\rho\frac{\partial S_1}{\partial\omega}\right]-\rho(\pi+S_1)\left[\frac{\partial\rho}{\partial\omega}\cos\theta-\frac{\partial\sigma}{\partial\omega}\sin\theta\right]\right\};$$

$$\frac{\partial r_4}{\partial\omega} = -\frac{1}{\sin\theta}\left\{\frac{\partial\rho}{\partial\omega}(\pi+\bar{\ell}+S_2)+\rho\left(\frac{\partial\bar{\ell}}{\partial\omega}+\frac{\partial S_2}{\partial\omega}\right)\right\};$$

$$\frac{\partial\tau}{\partial\omega}=0; \qquad \frac{\partial\sigma}{\partial\omega}=\frac{ae\sin\omega\cos I}{\beta}; \qquad \frac{\partial\rho}{\partial\omega}=\frac{1}{2a'[det(\mathcal{A})]^{\frac{1}{2}}}\frac{\partial}{\partial\omega}\big(det(\mathcal{A})\big);$$

$$\frac{\partial\bar{\ell}}{\partial\omega}=-\frac{\partial k}{\partial\omega}; \qquad \frac{\partial\bar{\ell}'}{\partial\omega}=0\,.$$

B.1.5 Derivatives of $\int(1/\mathrm{d}_1)\,d\ell d\ell'$

We write down the formulas needed for the computation of the derivatives of $\int_{\mathbb{T}^2}\frac{1}{\mathrm{d}_1}\,d\ell d\ell'$. The contents of this section has been used to write the FORTRAN programs to compute the weak averaged solutions.

DERIVATIVE WITH RESPECT TO e

$$\begin{aligned}\frac{\partial}{\partial e}\int_{\tau_{\pi,0}(\mathbb{T}^2)}\frac{1}{\mathrm{d}_1}\,d\ell d\ell' \;&=\; \frac{\partial}{\partial e}\left[\frac{1}{\sqrt{det(\mathcal{A}_d)}}\right]\cdot\left\{\sum_{j=1}^{4}\int_{\theta_j^-}^{\theta_{j+1}^-}\sqrt{(d^-_{min})^2+(r^-_j)^2(\theta)}\,d\theta-2\pi d^-_{min}\right\}+\\ &+\left[\frac{1}{\sqrt{det(\mathcal{A}_d)}}\right]\cdot\left\{\frac{1}{2}\sum_{j=1}^{4}\int_{\theta_j^-}^{\theta_{j+1}^-}\frac{\frac{\partial}{\partial e}[(d^-_{min})^2+(r^-_j)^2(\theta)]}{\sqrt{(d^-_{min})^2+(r^-_j)^2(\theta)}}\,d\theta-2\pi\frac{\partial}{\partial e}d^-_{min}\right\}.\end{aligned}$$

where $r_j^-(\theta)$, with $j=1\ldots4$, represent the lines r_j^- delimiting $\Xi_1[\overline{\mathbb{T}}^2]$ in polar coordinates:

$$r_1^-(\theta)=\frac{\rho_1\tau_1(\pi-S_1^-)}{\rho_1\cos\theta-\sigma_1\sin\theta}; \qquad r_2^-(\theta)=\frac{\rho_1(\pi-\bar{\ell}_1-S_2^-)}{\sin\theta};$$

$$r_3^-(\theta)=\frac{-\rho_1\tau_1(\pi+S_1^-)}{\rho_1\cos\theta-\sigma_1\sin\theta}; \qquad r_4^-(\theta)=\frac{-\rho_1(\pi+\bar{\ell}_1+S_2^-)}{\sin\theta};$$

while $\theta_1^-=\theta_5^--2\pi$ and θ_l^-, with $l=2\ldots5$, are the angles defined by

$$0<\theta_2^-<\theta_3^-<\pi<\theta_4^-<\theta_5^-<2\pi;$$

$$\tan\theta_2^-=\frac{\rho_1(\pi-\bar{\ell}_1-S_2^-)}{\sigma_1(\pi-\bar{\ell}_1-S_2^-)+\tau_1(\pi-S_1^-)}; \quad \tan\theta_3^-=\frac{\rho_1(\pi-\bar{\ell}_1-S_2^-)}{\sigma_1(\pi-\bar{\ell}_1-S_2^-)-\tau_1(\pi+S_1^-)};$$

$$\tan\theta_4^- = \frac{\rho_1(\pi+\overline{\ell}_1+S_2^-)}{\sigma_1(\pi+\overline{\ell}_1+S_2^-)+\tau_1(\pi+S_1^-)}; \quad \tan\theta_5^- = \frac{\rho_1(\pi+\overline{\ell}_1+S_2^-)}{\sigma_1(\pi+\overline{\ell}_1+S_2^-)-\tau_1(\pi-S_1^-)} \,.$$

The derivatives appearing in the previous formula are the following:

$$\frac{\partial}{\partial e}\left[\frac{1}{\sqrt{det(\mathcal{A}_d)}}\right] = -\frac{1}{2[det(\mathcal{A}_d)]^{3/2}}\frac{\partial}{\partial e}\left(det(\mathcal{A}_d)\right);$$

$$\frac{\partial}{\partial e}\left[(d_{min}^-)^2\right] = 2d_{nod}^-\frac{\partial}{\partial e}\left(d_{nod}^-\right)\left[\frac{a'^2\mathcal{G}_1^2\sin^2 I}{det(\mathcal{A}_d)}\right] + (d_{nod}^-)^2\frac{a'^2\sin^2 I}{det(\mathcal{A}_d)^2}\left[2det(\mathcal{A}_d)\mathcal{G}_1\frac{\partial\mathcal{G}_1}{\partial e} - \mathcal{G}_1^2\frac{\partial}{\partial e}\left(det(\mathcal{A}_d)\right)\right];$$

$$\frac{\partial}{\partial e}(det(\mathcal{A}_d)) = 2a'^2\mathcal{F}_1\frac{\partial\mathcal{F}_1}{\partial e} + 2a'^2\mathcal{G}_1\frac{\partial\mathcal{G}_1}{\partial e}\sin^2 I;$$

$$\frac{\partial S_1^-}{\partial e} = \frac{a'\cos I}{(det(\mathcal{A}_d))^2}\left\{det(\mathcal{A}_d)\left[\frac{\partial\mathcal{F}_1}{\partial e}\mathcal{G}_1 d_{nod}^- + \mathcal{F}_1\frac{\partial\mathcal{G}_1}{\partial e}d_{nod}^- + \mathcal{F}_1\mathcal{G}_1\frac{\partial}{\partial e}\left(d_{nod}^-\right)\right] - \mathcal{F}_1\mathcal{G}_1 d_{nod}^-\frac{\partial}{\partial e}\left(det(\mathcal{A}_d)\right)\right\};$$

$$\frac{\partial S_2^-}{\partial e} = -\frac{a'^2}{(det(\mathcal{A}_d))^2}\left\{det(\mathcal{A}_d)\left[\frac{\partial\mathcal{F}_1}{\partial e}d_{nod}^- + \mathcal{F}_1\frac{\partial}{\partial e}\left(d_{nod}^-\right)\right] - \mathcal{F}_1 d_{nod}^-\frac{\partial}{\partial e}\left(det(\mathcal{A}_d)\right)\right\};$$

$$\begin{aligned}\frac{\partial r_1^-}{\partial e} = \frac{\tau_1}{(\rho_1\cos\theta-\sigma_1\sin\theta)^2}\Bigg\{&(\rho_1\cos\theta-\sigma_1\sin\theta)\left[\frac{\partial\rho_1}{\partial e}(\pi-S_1^-) - \rho_1\frac{\partial S_1^-}{\partial e}\right] - \\ &-\rho_1(\pi-S_1^-)\left[\frac{\partial\rho_1}{\partial e}\cos\theta - \frac{\partial\sigma_1}{\partial e}\sin\theta\right]\Bigg\};\end{aligned}$$

$$\frac{\partial r_2^-}{\partial e} = \frac{1}{\sin\theta}\left\{\frac{\partial\rho_1}{\partial e}(\pi-\overline{\ell}_1-S_2^-) - \rho_1\left(\frac{\partial\overline{\ell}_1}{\partial e}+\frac{\partial S_2^-}{\partial e}\right)\right\};$$

$$\begin{aligned}\frac{\partial r_3^-}{\partial e} = -\frac{\tau_1}{(\rho_1\cos\theta-\sigma_1\sin\theta)^2}\Bigg\{&(\rho_1\cos\theta-\sigma_1\sin\theta)\left[\frac{\partial\rho_1}{\partial e}(\pi+S_1^-) + \rho_1\frac{\partial S_1^-}{\partial e}\right] - \\ &-\rho_1(\pi+S_1^-)\left[\frac{\partial\rho_1}{\partial e}\cos\theta - \frac{\partial\sigma_1}{\partial e}\sin\theta\right]\Bigg\};\end{aligned}$$

$$\frac{\partial r_4^-}{\partial e} = -\frac{1}{\sin\theta}\left\{\frac{\partial\rho_1}{\partial e}(\pi+\overline{\ell}_1+S_2^-) + \rho_1\left(\frac{\partial\overline{\ell}_1}{\partial e}+\frac{\partial S_2^-}{\partial e}\right)\right\};$$

$$\frac{\partial\tau_1}{\partial e} = 0; \qquad \frac{\partial\sigma_1}{\partial e} = -\frac{a\cos I}{\beta^3}(e-\cos\omega); \qquad \frac{\partial\rho_1}{\partial e} = \frac{1}{2a'[det(\mathcal{A}_d)]^{\frac{1}{2}}}\frac{\partial}{\partial e}\big(det(\mathcal{A}_d)\big);$$

$$\frac{\partial\overline{\ell}_1}{\partial e} = -\frac{\partial k_1}{\partial e}; \qquad \frac{\partial\overline{\ell}_1'}{\partial e} = 0\,.$$

DERIVATIVE WITH RESPECT TO ω

$$\begin{aligned}\frac{\partial}{\partial\omega}\int_{\tau_{\pi,0}(\mathbb{T}^2)}\frac{1}{\mathrm{d}_1}\,d\ell\, d\ell' &= \frac{\partial}{\partial\omega}\left[\frac{1}{\sqrt{det(\mathcal{A}_d)}}\right]\cdot\left\{\sum_{j=1}^{4}\int_{\theta_j^-}^{\theta_{j+1}^-}\sqrt{(d_{min}^-)^2+(r_j^-)^2(\theta)}\,d\theta-2\pi d_{min}^-\right\}+\\ &+\left[\frac{1}{\sqrt{det(\mathcal{A}_d)}}\right]\cdot\left\{\frac{1}{2}\sum_{j=1}^{4}\int_{\theta_j^-}^{\theta_{j+1}^-}\frac{\frac{\partial}{\partial\omega}[(d_{min}^-)^2+(r_j^-)^2(\theta)]}{\sqrt{(d_{min}^-)^2+(r_j^-)^2(\theta)}}\,d\theta-2\pi\frac{\partial}{\partial\omega}d_{min}^-\right\}\;.\end{aligned}$$

The derivatives appearing in the previous formula are the following:

$$\frac{\partial}{\partial\omega}\left[\frac{1}{\sqrt{det(\mathcal{A}_d)}}\right]=-\frac{1}{2[det(\mathcal{A}_d)]^{3/2}}\frac{\partial}{\partial\omega}\left(det(\mathcal{A}_d)\right)\;;$$

$$\frac{\partial}{\partial\omega}\left[(d_{min}^-)^2\right]=2d_{nod}^-\frac{\partial}{\partial\omega}\left(d_{nod}^-\right)\left[\frac{a'^2\mathcal{G}_1^2\sin^2 I}{det(\mathcal{A}_d)}\right]+(d_{nod}^-)^2\frac{a'^2\sin^2 I}{det(\mathcal{A}_d)^2}\left[2det(\mathcal{A}_d)\mathcal{G}_1\frac{\partial\mathcal{G}_1}{\partial\omega}-\mathcal{G}_1^2\frac{\partial}{\partial\omega}\left(det(\mathcal{A}_d)\right)\right]\;;$$

$$\frac{\partial}{\partial\omega}(det(\mathcal{A}_d))=2a'^2\mathcal{F}_1\frac{\partial\mathcal{F}_1}{\partial\omega}+2a'^2\mathcal{G}_1\frac{\partial\mathcal{G}_1}{\partial\omega}\sin^2 I;$$

$$\frac{\partial S_1^-}{\partial\omega}=\frac{a'\cos I}{(det(\mathcal{A}_d))^2}\left\{det(\mathcal{A}_d)\left[\frac{\partial\mathcal{F}_1}{\partial\omega}\mathcal{G}_1 d_{nod}^-+\mathcal{F}_1\frac{\partial\mathcal{G}_1}{\partial\omega}d_{nod}^-+\mathcal{F}_1\mathcal{G}_1\frac{\partial}{\partial\omega}\left(d_{nod}^-\right)\right]-\mathcal{F}_1\mathcal{G}_1 d_{nod}^-\frac{\partial}{\partial\omega}\left(det(\mathcal{A}_d)\right)\right\}\;;$$

$$\frac{\partial S_2^-}{\partial\omega}=-\frac{a'^2}{(det(\mathcal{A}_d))^2}\left\{det(\mathcal{A}_d)\left[\frac{\partial\mathcal{F}_1}{\partial\omega}d_{nod}^-+\mathcal{F}_1\frac{\partial}{\partial\omega}\left(d_{nod}^-\right)\right]-\mathcal{F}_1 d_{nod}^-\frac{\partial}{\partial\omega}\left(det(\mathcal{A}_d)\right)\right\}\;;$$

$$\begin{aligned}\frac{\partial r_1^-}{\partial\omega}=\frac{\tau_1}{(\rho_1\cos\theta-\sigma_1\sin\theta)^2}\Bigg\{&(\rho_1\cos\theta-\sigma_1\sin\theta)\left[\frac{\partial\rho_1}{\partial\omega}(\pi-S_1^-)-\rho_1\frac{\partial S_1^-}{\partial\omega}\right]-\\ &-\rho_1(\pi-S_1^-)\left[\frac{\partial\rho_1}{\partial\omega}\cos\theta-\frac{\partial\sigma_1}{\partial\omega}\sin\theta\right]\Bigg\};\end{aligned}$$

$$\frac{\partial r_2^-}{\partial\omega}=\frac{1}{\sin\theta}\left\{\frac{\partial\rho_1}{\partial\omega}(\pi-\overline{\ell}_1-S_2^-)-\rho_1\left(\frac{\partial\overline{\ell}_1}{\partial\omega}+\frac{\partial S_2^-}{\partial\omega}\right)\right\}\;;$$

$$\begin{aligned}\frac{\partial r_3^-}{\partial\omega}=-\frac{\tau_1}{(\rho_1\cos\theta-\sigma_1\sin\theta)^2}\Bigg\{&(\rho_1\cos\theta-\sigma_1\sin\theta)\left[\frac{\partial\rho_1}{\partial\omega}(\pi+S_1^-)+\rho_1\frac{\partial S_1^-}{\partial\omega}\right]-\\ &-\rho_1(\pi+S_1^-)\left[\frac{\partial\rho_1}{\partial\omega}\cos\theta-\frac{\partial\sigma_1}{\partial\omega}\sin\theta\right]\Bigg\};\end{aligned}$$

$$\frac{\partial r_4^-}{\partial\omega}=-\frac{1}{\sin\theta}\left\{\frac{\partial\rho_1}{\partial\omega}(\pi+\overline{\ell}_1+S_2^-)+\rho_1\left(\frac{\partial\overline{\ell}_1}{\partial\omega}+\frac{\partial S_2^-}{\partial\omega}\right)\right\}\;;$$

$$\frac{\partial\tau_1}{\partial\omega}=0;\qquad\frac{\partial\sigma_1}{\partial\omega}=-\frac{ae\sin\omega\cos I}{\beta}\;;\qquad\frac{\partial\rho_1}{\partial\omega}=\frac{1}{2a'[det(\mathcal{A}_d)]^{\frac{1}{2}}}\frac{\partial}{\partial\omega}\big(det(\mathcal{A}_d)\big)\;;$$

$$\frac{\partial\overline{\ell}_1}{\partial\omega}=-\frac{\partial k_1}{\partial\omega}\;;\qquad\frac{\partial\overline{\ell}_1'}{\partial\omega}=0\;.$$

Appendix C

C.1 Elliptic inclined case

In this section we give the explicit formulas for the derivatives of the direct term of the perturbing function and for the approximations of this function at the ascending and descending node defined by the Wetherill functions. The framework is the one of the general case with the planets on their elliptic inclined orbits.

C.1.1 Derivatives of $1/\mathfrak{D}$

We recall that the square of the distance $\mathfrak{D}$ is defined by

$$\begin{aligned}\mathfrak{D}^2(u,u') &= a^2(1-e\cos u)^2 + a'^2(1-e'\cos u')^2 - \\ &\quad -2aa'\{[(\cos u - e)\cos\omega_M - \beta\sin u\sin\omega_M][(\cos u' - e')\cos\omega'_M - \beta'\sin u'\sin\omega'_M] + \\ &\quad +\cos I_M[(\cos u - e)\sin\omega_M + \beta\sin u\cos\omega_M][(\cos u' - e')\sin\omega'_M + \beta'\sin u'\cos\omega'_M]\}\ .\end{aligned}$$

DERIVATIVE WITH RESPECT TO e

$$\frac{\partial}{\partial e}\left(\frac{1}{\mathfrak{D}}\right) = -\frac{1}{2\mathfrak{D}^3}\frac{\partial}{\partial e}\mathfrak{D}^2$$

$$\begin{aligned}\frac{\partial}{\partial e}\mathfrak{D}^2 &= a^2[-2(\cos u - e) - 2e\sin^2 u] - \\ &\quad -2aa'\left\{[-\cos\omega_M - \left[\frac{-e}{\beta}\right]\sin u\sin\omega_M][(\cos u' - e')\cos\omega'_M - \beta'\sin u'\sin\omega'_M] + \right.\\ &\quad \left. +\cos I_M\left[-\sin\omega_M + \left[\frac{-e}{\beta}\right]\sin u\cos\omega_M\right][(\cos u' - e')\sin\omega'_M + \beta'\sin u'\cos\omega'_M]\right\}\end{aligned}$$

DERIVATIVE WITH RESPECT TO ω_M

$$\frac{\partial}{\partial\omega_M}\left(\frac{1}{\mathfrak{D}}\right) = -\frac{1}{2\mathfrak{D}^3}\frac{\partial}{\partial\omega_M}\mathfrak{D}^2$$

$$\frac{\partial}{\partial\omega_M}\mathfrak{D}^2 = -2aa'\{[-\sin\omega_M(\cos u - e) - \beta\sin u\cos\omega_M][(\cos u' - e')\cos\omega'_M - \beta'\sin u'\sin\omega'_M] +$$

$$+\cos I_M[\cos\omega_M(\cos u - e) - \beta\sin u\sin\omega_M][(\cos u' - e')\sin\omega'_M + \beta'\sin u'\cos\omega'_M]\}$$

DERIVATIVE WITH RESPECT TO ω'_M

$$\frac{\partial}{\partial\omega'_M}\left(\frac{1}{\mathfrak{D}}\right) = -\frac{1}{2\mathfrak{D}^3}\frac{\partial}{\partial\omega'_M}\mathfrak{D}^2$$

$$\begin{aligned}\frac{\partial}{\partial\omega'_M}\mathfrak{D}^2 &= -2aa'\{[(\cos u - e)\cos\omega_M - \beta\sin u\sin\omega_M][-\sin\omega'_M(\cos u' - e') - \beta\sin u'\cos\omega'_M] + \\ &\quad +\cos I_M[(\cos u - e)\sin\omega_M + \beta\sin u\cos\omega_M][\cos\omega'_M(\cos u' - e') - \beta'\sin u'\sin\omega'_M]\}\end{aligned}$$

DERIVATIVE WITH RESPECT TO I_M

$$\frac{\partial}{\partial I_M}\left(\frac{1}{\mathfrak{D}}\right) = -\frac{1}{2\mathfrak{D}^3}\frac{\partial}{\partial I_M}\mathfrak{D}^2$$

$$\frac{\partial}{\partial I_M}\mathfrak{D}^2 = 2aa'\sin I_M\,[(\cos u - e)\sin\omega + \beta\sin u\cos\omega]\,\left[(\cos u' - e')\sin\omega' + \beta'\sin u'\cos\omega'\right]$$

C.1.2 Derivatives of $1/\mathfrak{d}$

We recall that the square of the Wetherill function, for the approximation of the distance at the ascending node, is given by

$$\begin{aligned}\mathfrak{d}^2(u,u') &= k'^2[\mathcal{F}'^2 + \mathcal{G}'^2] + k^2[\mathcal{F}^2 + \mathcal{G}^2] - 2kk'[\mathcal{F}\mathcal{F}' + \mathcal{G}\mathcal{G}'\cos I_M] - \\ &\quad -2d^+_{nod}[\mathcal{F}k - \mathcal{F}'k'] + (d^+_{nod})^2\end{aligned}$$

where

$$\begin{cases} k = \ell - \overline{\ell} = u - \overline{u} - e(\sin u - \sin\overline{u}) \\ k' = \ell' - \overline{\ell}' = u' - \overline{u}' - e'(\sin u' - \sin\overline{u}') \end{cases}$$

DERIVATIVE WITH RESPECT TO e

$$\frac{\partial}{\partial e}\left(\frac{1}{\mathfrak{d}}\right) = -\frac{1}{2\mathfrak{d}^3}\frac{\partial}{\partial e}\mathfrak{d}^2$$

$$\begin{aligned}\frac{\partial}{\partial e}\mathfrak{d}^2 &= 2k'\frac{\partial k'}{\partial e}[\mathcal{F}'^2 + \mathcal{G}'^2] + k'^2\frac{\partial}{\partial e}\left([\mathcal{F}'^2 + \mathcal{G}'^2]\right) + 2k\frac{\partial k}{\partial e}[\mathcal{F}^2 + \mathcal{G}^2] + k^2\frac{\partial}{\partial e}\left([\mathcal{F}^2 + \mathcal{G}^2]\right) - \\ &\quad -2\frac{\partial k}{\partial e}k'[\mathcal{F}\mathcal{F}' + \mathcal{G}\mathcal{G}'\cos I_M] - 2k\frac{\partial k'}{\partial e}[\mathcal{F}\mathcal{F}' + \mathcal{G}\mathcal{G}'\cos I_M] - 2kk'\frac{\partial}{\partial e}\left([\mathcal{F}\mathcal{F}' + \mathcal{G}\mathcal{G}'\cos I_M]\right) - \\ &\quad -2\frac{\partial}{\partial e}\left(d^+_{nod}\right)[\mathcal{F}k - \mathcal{F}'k'] - 2d^+_{nod}\frac{\partial}{\partial e}\left([\mathcal{F}k - \mathcal{F}'k']\right) + 2d^+_{nod}\frac{\partial}{\partial e}\left(d^+_{nod}\right)\end{aligned}$$

in which

$$\frac{\partial k}{\partial e} = -(1 - e\cos\overline{u})\frac{\partial\overline{u}}{\partial e} - (\sin u - \sin\overline{u}); \qquad \frac{\partial\overline{u}}{\partial e} = \frac{\sin\omega_M}{\beta(1 + e\cos\omega_M)}; \qquad \frac{\partial k'}{\partial e} = 0;$$

$$\frac{\partial}{\partial e}\left([\mathcal{F}^2+\mathcal{G}^2]\right)=2\left[\mathcal{F}\frac{\partial \mathcal{F}}{\partial e}+\mathcal{G}\frac{\partial \mathcal{G}}{\partial e}\right]; \qquad \frac{\partial}{\partial e}\left([\mathcal{F}'^2+\mathcal{G}'^2]\right)=0;$$
$$\frac{\partial}{\partial e}\left([\mathcal{F}\mathcal{F}'+\mathcal{G}\mathcal{G}'\cos I_M]\right)=\left[\frac{\partial \mathcal{F}}{\partial e}\mathcal{F}'+\mathcal{F}\frac{\partial \mathcal{F}'}{\partial e}+\left(\frac{\partial \mathcal{G}}{\partial e}\mathcal{G}'+\mathcal{G}\frac{\partial \mathcal{G}'}{\partial e}\right)\cos I_M\right];$$
$$\frac{\partial}{\partial e}\left([\mathcal{F}k-\mathcal{F}'k']\right)=\frac{\partial \mathcal{F}}{\partial e}k+\mathcal{F}\frac{\partial k}{\partial e}-\left(\frac{\partial \mathcal{F}'}{\partial e}k'+\mathcal{F}'\frac{\partial k'}{\partial e}\right)\Big\};$$
$$\frac{\partial \mathcal{F}}{\partial e}=\frac{a\sin\omega_M}{\beta^3}; \qquad \frac{\partial \mathcal{G}}{\partial e}=\frac{a(\cos\omega_M+e)}{\beta^3}; \qquad \frac{\partial \mathcal{F}'}{\partial e}=0; \qquad \frac{\partial \mathcal{G}'}{\partial e}=0;$$
$$\frac{\partial}{\partial e}\left(d^+_{nod}\right)=-\frac{a[\cos\omega_M(1+e^2)+2e]}{(1+e\cos\omega_M)^2}.$$

DERIVATIVE WITH RESPECT TO ω_M

$$\frac{\partial}{\partial \omega_M}\left(\frac{1}{\eth}\right)=-\frac{1}{2\eth^3}\frac{\partial}{\partial \omega_M}\eth^2$$

$$\begin{aligned}\frac{\partial}{\partial \omega_M}\eth^2 \;=\;& 2k'\frac{\partial k'}{\partial \omega_M}[\mathcal{F}'^2+\mathcal{G}'^2]+k'^2\frac{\partial}{\partial \omega_M}\left([\mathcal{F}'^2+\mathcal{G}'^2]\right)+2k\frac{\partial k}{\partial \omega_M}[\mathcal{F}^2+\mathcal{G}^2]+k^2\frac{\partial}{\partial \omega_M}\left([\mathcal{F}^2+\mathcal{G}^2]\right)-\\ &-2\frac{\partial k}{\partial \omega_M}k'[\mathcal{F}\mathcal{F}'+\mathcal{G}\mathcal{G}'\cos I_M]-2k\frac{\partial k'}{\partial \omega_M}[\mathcal{F}\mathcal{F}'+\mathcal{G}\mathcal{G}'\cos I_M]-2kk'\frac{\partial}{\partial \omega_M}\left([\mathcal{F}\mathcal{F}'+\mathcal{G}\mathcal{G}'\cos I_M]\right)-\\ &-2\frac{\partial}{\partial \omega_M}\left(d^+_{nod}\right)[\mathcal{F}k-\mathcal{F}'k']-2d^+_{nod}\frac{\partial}{\partial \omega_M}\left([\mathcal{F}k-\mathcal{F}'k']\right)+2d^+_{nod}\frac{\partial}{\partial \omega_M}\left(d^+_{nod}\right)\end{aligned}$$

in which

$$\frac{\partial k}{\partial \omega_M}=-(1-e\cos\overline{u})\frac{\partial \overline{u}}{\partial \omega_M}; \qquad \frac{\partial \overline{u}}{\partial \omega_M}=-\frac{\beta}{1+e\cos\omega_M}; \qquad \frac{\partial k'}{\partial \omega_M}=0;$$
$$\frac{\partial}{\partial \omega_M}\left([\mathcal{F}^2+\mathcal{G}^2]\right)=2\left[\mathcal{F}\frac{\partial \mathcal{F}}{\partial \omega_M}+\mathcal{G}\frac{\partial \mathcal{G}}{\partial \omega_M}\right]; \qquad \frac{\partial}{\partial \omega_M}\left([\mathcal{F}'^2+\mathcal{G}'^2]\right)=0;$$
$$\frac{\partial}{\partial \omega_M}\left([\mathcal{F}\mathcal{F}'+\mathcal{G}\mathcal{G}'\cos I_M]\right)=\left[\frac{\partial \mathcal{F}}{\partial \omega_M}\mathcal{F}'+\mathcal{F}\frac{\partial \mathcal{F}'}{\partial \omega_M}+\left(\frac{\partial \mathcal{G}}{\partial \omega_M}\mathcal{G}'+\mathcal{G}\frac{\partial \mathcal{G}'}{\partial \omega_M}\right)\cos I_M\right];$$
$$\frac{\partial}{\partial \omega_M}\left([\mathcal{F}k-\mathcal{F}'k']\right)=\frac{\partial \mathcal{F}}{\partial \omega_M}k+\mathcal{F}\frac{\partial k}{\partial \omega_M}-\left(\frac{\partial \mathcal{F}'}{\partial \omega_M}k'+\mathcal{F}'\frac{\partial k'}{\partial \omega_M}\right)\Big\};$$
$$\frac{\partial \mathcal{F}}{\partial \omega_M}=\frac{ae\cos\omega_M}{\beta}; \qquad \frac{\partial \mathcal{G}}{\partial \omega_M}=-\frac{ae\sin\omega_M}{\beta}; \qquad \frac{\partial \mathcal{F}'}{\partial \omega_M}=0; \qquad \frac{\partial \mathcal{G}'}{\partial \omega_M}=0;$$
$$\frac{\partial}{\partial \omega_M}\left(d^+_{nod}\right)=\frac{ae\beta^2\sin\omega_M}{(1+e\cos\omega_M)^2}.$$

DERIVATIVE WITH RESPECT TO ω'_M

$$\frac{\partial}{\partial \omega'_M}\left(\frac{1}{\eth}\right)=-\frac{1}{2\eth^3}\frac{\partial}{\partial \omega'_M}\eth^2$$

$$\frac{\partial}{\partial \omega'_M}\eth^2 \;=\; 2k'\frac{\partial k'}{\partial \omega'_M}[\mathcal{F}'^2+\mathcal{G}'^2]+k'^2\frac{\partial}{\partial \omega'_M}\left([\mathcal{F}'^2+\mathcal{G}'^2]\right)+2k\frac{\partial k}{\partial \omega'_M}[\mathcal{F}^2+\mathcal{G}^2]+$$

$$+k^2\frac{\partial}{\partial\omega'_M}\left([\mathcal{F}^2+\mathcal{G}^2]\right)-2\frac{\partial k}{\partial\omega'_M}k'[\mathcal{F}\mathcal{F}'+\mathcal{G}\mathcal{G}'\cos I_M]-2k\frac{\partial k'}{\partial\omega'_M}[\mathcal{F}\mathcal{F}'+\mathcal{G}\mathcal{G}'\cos I_M]-$$
$$-2kk'\frac{\partial}{\partial\omega'_M}\left([\mathcal{F}\mathcal{F}'+\mathcal{G}\mathcal{G}'\cos I_M]\right)-2\frac{\partial}{\partial\omega'_M}\left(d^+_{nod}\right)[\mathcal{F}k-\mathcal{F}'k']-$$
$$-2d^+_{nod}\frac{\partial}{\partial\omega'_M}\left([\mathcal{F}k-\mathcal{F}'k']\right)+2d^+_{nod}\frac{\partial}{\partial\omega'_M}\left(d^+_{nod}\right)$$

in which

$$\frac{\partial k}{\partial\omega'_M}=0;\qquad \frac{\partial k'}{\partial\omega'_M}=-(1-e'\cos\overline{u}')\frac{\partial\overline{u}'}{\partial\omega'_M};\qquad \frac{\partial\overline{u}'}{\partial\omega'_M}=-\frac{\beta'}{1+e'\cos\omega'_M};$$
$$\frac{\partial}{\partial\omega'_M}\left([\mathcal{F}^2+\mathcal{G}^2]\right)=0;\qquad \frac{\partial}{\partial\omega'_M}\left([\mathcal{F}'^2+\mathcal{G}'^2]\right)=2\left[\mathcal{F}'\frac{\partial\mathcal{F}'}{\partial\omega'_M}+\mathcal{G}'\frac{\partial\mathcal{G}'}{\partial\omega'_M}\right];$$
$$\frac{\partial}{\partial\omega'_M}\left([\mathcal{F}\mathcal{F}'+\mathcal{G}\mathcal{G}'\cos I_M]\right)=\left[\frac{\partial\mathcal{F}}{\partial\omega'_M}\mathcal{F}'+\mathcal{F}\frac{\partial\mathcal{F}'}{\partial\omega'_M}+\left(\frac{\partial\mathcal{G}}{\partial\omega'_M}\mathcal{G}'+\mathcal{G}\frac{\partial\mathcal{G}'}{\partial\omega'_M}\right)\cos I_M\right];$$
$$\frac{\partial}{\partial\omega'_M}\left([\mathcal{F}k-\mathcal{F}'k']\right)=\frac{\partial\mathcal{F}}{\partial\omega'_M}k+\mathcal{F}\frac{\partial k}{\partial\omega'_M}-\left(\frac{\partial\mathcal{F}'}{\partial\omega'_M}k'+\mathcal{F}'\frac{\partial k'}{\partial\omega'_M}\right)\Bigg\};$$
$$\frac{\partial\mathcal{F}}{\partial\omega'_M}=0;\qquad \frac{\partial\mathcal{G}}{\partial\omega'_M}=0;\qquad \frac{\partial\mathcal{F}'}{\partial\omega'_M}=\frac{a'e'\cos\omega'_M}{\beta'};\qquad \frac{\partial\mathcal{G}'}{\partial\omega'_M}=\frac{-a'e'\sin\omega'_M}{\beta'};$$
$$\frac{\partial}{\partial\omega'_M}\left(d^+_{nod}\right)=-\frac{a'e'\beta'^2\sin\omega'_M}{(1+e'\cos\omega'_M)^2}.$$

C.1.3 Derivatives of $1/\mathfrak{d}_1$

We recall that the square of the Wetherill function, for the approximation of the distance at the descending node, is given by

$$\begin{aligned}\mathfrak{d}_1^2(u,u') &= k_1'^2[\mathcal{F}_1'^2+\mathcal{G}_1'^2]+k_1^2[\mathcal{F}_1^2+\mathcal{G}_1^2]-2k_1k_1'[\mathcal{F}_1\mathcal{F}_1'+\mathcal{G}_1\mathcal{G}_1'\cos I_M]+\\ &\quad +2d^-_{nod}[\mathcal{F}_1k_1-\mathcal{F}_1'k_1']+(d^-_{nod})^2\end{aligned}$$

where

$$\begin{cases}k_1=\ell-\overline{\ell}_1=u-\overline{u}_1-e(\sin u-\sin\overline{u}_1)\\ k_1'=\ell'-\overline{\ell}_1'=u'-\overline{u}_1'-e'(\sin u'-\sin\overline{u}_1')\end{cases}$$

DERIVATIVE WITH RESPECT TO e

$$\frac{\partial}{\partial e}\left(\frac{1}{\mathfrak{d}_1}\right)=-\frac{1}{2\mathfrak{d}_1^3}\frac{\partial}{\partial e}\mathfrak{d}_1^2$$

$$\begin{aligned}\frac{\partial}{\partial e}\mathfrak{d}_1^2 &= 2k_1'\frac{\partial k_1'}{\partial e}[\mathcal{F}_1'^2+\mathcal{G}_1'^2]+k_1'^2\frac{\partial}{\partial e}\left([\mathcal{F}_1'^2+\mathcal{G}_1'^2]\right)+2k_1\frac{\partial k_1}{\partial e}[\mathcal{F}_1^2+\mathcal{G}_1^2]+k_1^2\frac{\partial}{\partial e}\left([\mathcal{F}_1^2+\mathcal{G}_1^2]\right)-\\ &\quad -2\frac{\partial k_1}{\partial e}k_1'[\mathcal{F}_1\mathcal{F}_1'+\mathcal{G}_1\mathcal{G}_1'\cos I_M]-2k_1\frac{\partial k_1'}{\partial e}[\mathcal{F}_1\mathcal{F}_1'+\mathcal{G}_1\mathcal{G}_1'\cos I_M]-2k_1k_1'\frac{\partial}{\partial e}\left([\mathcal{F}_1\mathcal{F}_1'+\mathcal{G}_1\mathcal{G}_1'\cos I_M]\right)+\\ &\quad +2\frac{\partial}{\partial e}\left(d^-_{nod}\right)[\mathcal{F}_1k_1-\mathcal{F}_1'k_1']+2d^-_{nod}\frac{\partial}{\partial e}\left([\mathcal{F}_1k_1-\mathcal{F}_1'k_1']\right)+2d^-_{nod}\frac{\partial}{\partial e}\left(d^-_{nod}\right)\end{aligned}$$

in which

$$\frac{\partial k_1}{\partial e}=-(1-e\cos\overline{u}_1)\frac{\partial\overline{u}_1}{\partial e}-(\sin u-\sin\overline{u}_1);\qquad \frac{\partial\overline{u}_1}{\partial e}=-\frac{\sin\omega_M}{\beta(1-e\cos\omega_M)};\qquad \frac{\partial k_1'}{\partial e}=0;$$

$$\frac{\partial}{\partial e}\left([\mathcal{F}_1^2+\mathcal{G}_1^2]\right)=2\left[\mathcal{F}_1\frac{\partial \mathcal{F}_1}{\partial e}+\mathcal{G}_1\frac{\partial \mathcal{G}_1}{\partial e}\right]; \qquad \frac{\partial}{\partial e}\left([\mathcal{F}_1'^2+\mathcal{G}_1'^2]\right)=0;$$
$$\frac{\partial}{\partial e}\left([\mathcal{F}_1\mathcal{F}_1'+\mathcal{G}_1\mathcal{G}_1'\cos I_M]\right)=\left[\frac{\partial \mathcal{F}_1}{\partial e}\mathcal{F}_1'+\mathcal{F}_1\frac{\partial \mathcal{F}_1'}{\partial e}+\left(\frac{\partial \mathcal{G}_1}{\partial e}\mathcal{G}_1'+\mathcal{G}_1\frac{\partial \mathcal{G}_1'}{\partial e}\right)\cos I_M\right];$$
$$\frac{\partial}{\partial e}\left([\mathcal{F}_1k_1-\mathcal{F}_1'k_1']\right)=\frac{\partial \mathcal{F}_1}{\partial e}k_1+\mathcal{F}_1\frac{\partial k_1}{\partial e}-\left(\frac{\partial \mathcal{F}_1'}{\partial e}k_1'+\mathcal{F}_1'\frac{\partial k_1'}{\partial e}\right)\Bigg\};$$
$$\frac{\partial \mathcal{F}_1}{\partial e}=\frac{a\sin\omega_M}{\beta^3}; \qquad \frac{\partial \mathcal{G}_1}{\partial e}=\frac{a(\cos\omega_M-e)}{\beta^3}; \qquad \frac{\partial \mathcal{F}_1'}{\partial e}=0; \qquad \frac{\partial \mathcal{G}_1'}{\partial e}=0;$$
$$\frac{\partial}{\partial e}\left(d_{nod}^-\right)=\frac{a[\cos\omega_M(1+e^2)-2e]}{(1-e\cos\omega_M)^2}.$$

DERIVATIVE WITH RESPECT TO ω_M

$$\frac{\partial}{\partial \omega_M}\left(\frac{1}{\eth_1}\right)=-\frac{1}{2\eth_1^3}\frac{\partial}{\partial \omega_M}\eth_1^2$$

$$\begin{aligned}\frac{\partial}{\partial \omega_M}\eth_1^2 \;=\;& 2k_1'\frac{\partial k_1'}{\partial \omega_M}[\mathcal{F}_1'^2+\mathcal{G}_1'^2]+k_1'^2\frac{\partial}{\partial \omega_M}\left([\mathcal{F}_1'^2+\mathcal{G}_1'^2]\right)+2k_1\frac{\partial k_1}{\partial \omega_M}[\mathcal{F}_1^2+\mathcal{G}_1^2]+\\
&+k_1^2\frac{\partial}{\partial \omega_M}\left([\mathcal{F}_1^2+\mathcal{G}_1^2]\right)-2\frac{\partial k_1}{\partial \omega_M}k_1'[\mathcal{F}_1\mathcal{F}_1'+\mathcal{G}_1\mathcal{G}_1'\cos I_M]-\\
&-2k_1\frac{\partial k_1'}{\partial \omega_M}[\mathcal{F}_1\mathcal{F}_1'+\mathcal{G}_1\mathcal{G}_1'\cos I_M]-2k_1k_1'\frac{\partial}{\partial \omega_M}\left([\mathcal{F}_1\mathcal{F}_1'+\mathcal{G}_1\mathcal{G}_1'\cos I_M]\right)+\\
&+2\frac{\partial}{\partial \omega_M}\left(d_{nod}^-\right)[\mathcal{F}_1k_1-\mathcal{F}_1'k_1']+2d_{nod}^-\frac{\partial}{\partial \omega_M}\left([\mathcal{F}_1k_1-\mathcal{F}_1'k_1']\right)+2d_{nod}^-\frac{\partial}{\partial \omega_M}\left(d_{nod}^-\right)\end{aligned}$$

in which

$$\frac{\partial k_1}{\partial \omega_M}=-(1-e\cos\overline{u}_1)\frac{\partial \overline{u}_1}{\partial \omega_M}; \qquad \frac{\partial \overline{u}_1}{\partial \omega_M}=-\frac{\beta}{1-e\cos\omega_M}; \qquad \frac{\partial k_1'}{\partial \omega_M}=0;$$
$$\frac{\partial}{\partial \omega_M}\left([\mathcal{F}_1^2+\mathcal{G}_1^2]\right)=2\left[\mathcal{F}_1\frac{\partial \mathcal{F}_1}{\partial \omega_M}+\mathcal{G}_1\frac{\partial \mathcal{G}_1}{\partial \omega_M}\right]; \qquad \frac{\partial}{\partial \omega_M}\left([\mathcal{F}_1'^2+\mathcal{G}_1'^2]\right)=0;$$
$$\frac{\partial}{\partial \omega_M}\left([\mathcal{F}_1\mathcal{F}_1'+\mathcal{G}_1\mathcal{G}_1'\cos I_M]\right)=\left[\frac{\partial \mathcal{F}_1}{\partial \omega_M}\mathcal{F}_1'+\mathcal{F}_1\frac{\partial \mathcal{F}_1'}{\partial \omega_M}+\left(\frac{\partial \mathcal{G}_1}{\partial \omega_M}\mathcal{G}_1'+\mathcal{G}_1\frac{\partial \mathcal{G}_1'}{\partial \omega_M}\right)\cos I_M\right];$$
$$\frac{\partial}{\partial \omega_M}\left([\mathcal{F}_1k_1-\mathcal{F}_1'k_1']\right)=\frac{\partial \mathcal{F}_1}{\partial \omega_M}k_1+\mathcal{F}_1\frac{\partial k_1}{\partial \omega_M}-\left(\frac{\partial \mathcal{F}_1'}{\partial \omega_M}k_1'+\mathcal{F}_1'\frac{\partial k_1'}{\partial \omega_M}\right)\Bigg\};$$
$$\frac{\partial \mathcal{F}_1}{\partial \omega_M}=\frac{ae\cos\omega_M}{\beta}; \qquad \frac{\partial \mathcal{G}_1}{\partial \omega_M}=-\frac{ae\sin\omega_M}{\beta}; \qquad \frac{\partial \mathcal{F}_1'}{\partial \omega_M}=0; \qquad \frac{\partial \mathcal{G}_1'}{\partial \omega_M}=0;$$
$$\frac{\partial}{\partial \omega_M}\left(d_{nod}^-\right)=-\frac{ae\beta^2\sin\omega_M}{(1-e\cos\omega_M)^2}.$$

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$$\frac{\partial}{\partial \omega_M'}\left(\frac{1}{\eth_1}\right)=-\frac{1}{2\eth_1^3}\frac{\partial}{\partial \omega_M'}\eth_1^2$$

$$
\begin{aligned}
\frac{\partial}{\partial\omega'_M}\mathfrak{d}_1^2 \;=\;& 2k'_1\frac{\partial k'_1}{\partial\omega'_M}[\mathcal{F}_1'^2+\mathcal{G}_1'^2]+k_1'^2\frac{\partial}{\partial\omega'_M}\left([\mathcal{F}_1'^2+\mathcal{G}_1'^2]\right)+\\
& 2k_1\frac{\partial k_1}{\partial\omega'_M}[\mathcal{F}_1^2+\mathcal{G}_1^2]+k_1^2\frac{\partial}{\partial\omega'_M}\left([\mathcal{F}_1^2+\mathcal{G}_1^2]\right)-2\frac{\partial k_1}{\partial\omega'_M}k'_1[\mathcal{F}_1\mathcal{F}'_1+\mathcal{G}_1\mathcal{G}'_1\cos I_M]-\\
& -2k_1\frac{\partial k'_1}{\partial\omega'_M}[\mathcal{F}_1\mathcal{F}'_1+\mathcal{G}_1\mathcal{G}'_1\cos I_M]-2k_1k'_1\frac{\partial}{\partial\omega'_M}\left([\mathcal{F}_1\mathcal{F}'_1+\mathcal{G}_1\mathcal{G}'_1\cos I_M]\right)+\\
& +2\frac{\partial}{\partial\omega'_M}\left(d^-_{nod}\right)[\mathcal{F}_1k_1-\mathcal{F}'_1k'_1]+2d^-_{nod}\frac{\partial}{\partial\omega'_M}\left([\mathcal{F}_1k_1-\mathcal{F}'_1k'_1]\right)+2d^-_{nod}\frac{\partial}{\partial\omega'_M}\left(d^-_{nod}\right)
\end{aligned}
$$

in which

$$
\begin{aligned}
&\frac{\partial k_1}{\partial\omega'_M}=0; \qquad \frac{\partial k'_1}{\partial\omega'_M}=-(1-e'\cos\overline{u}'_1)\frac{\partial\overline{u}'_1}{\partial\omega'_M}; \qquad \frac{\partial\overline{u}'_1}{\partial\omega'_M}=-\frac{\beta'}{1-e'\cos\omega'_M};\\
&\frac{\partial}{\partial\omega'_M}\left([\mathcal{F}_1^2+\mathcal{G}_1^2]\right)=0; \qquad \frac{\partial}{\partial\omega'_M}\left([\mathcal{F}_1'^2+\mathcal{G}_1'^2]\right)=2\left[\mathcal{F}'_1\frac{\partial\mathcal{F}'_1}{\partial\omega'_M}+\mathcal{G}'_1\frac{\partial\mathcal{G}'_1}{\partial\omega'_M}\right];\\
&\frac{\partial}{\partial\omega'_M}\left([\mathcal{F}_1\mathcal{F}'_1+\mathcal{G}_1\mathcal{G}'_1\cos I_M]\right)=\left[\frac{\partial\mathcal{F}_1}{\partial\omega'_M}\mathcal{F}'_1+\mathcal{F}_1\frac{\partial\mathcal{F}'_1}{\partial\omega'_M}+\left(\frac{\partial\mathcal{G}_1}{\partial\omega'_M}\mathcal{G}'_1+\mathcal{G}_1\frac{\partial\mathcal{G}'_1}{\partial\omega'_M}\right)\cos I_M\right];\\
&\frac{\partial}{\partial\omega'_M}\left([\mathcal{F}_1k_1-\mathcal{F}'_1k'_1]\right)=\frac{\partial\mathcal{F}_1}{\partial\omega'_M}k_1+\mathcal{F}_1\frac{\partial k_1}{\partial\omega'_M}-\left(\frac{\partial\mathcal{F}'_1}{\partial\omega'_M}k'_1+\mathcal{F}'_1\frac{\partial k'_1}{\partial\omega'_M}\right)\Big\};\\
&\frac{\partial\mathcal{F}_1}{\partial\omega'_M}=0; \qquad \frac{\partial\mathcal{G}_1}{\partial\omega'_M}=0; \qquad \frac{\partial\mathcal{F}'_1}{\partial\omega'_M}=\frac{a'e'\cos\omega'_M}{\beta'}; \qquad \frac{\partial\mathcal{G}'_1}{\partial\omega'_M}=-\frac{a'e'\sin\omega'_M}{\beta'};\\
&\frac{\partial}{\partial\omega'_M}\left(d^-_{nod}\right)=\frac{a'e'\beta'^2\sin\omega'_M}{(1-e'\cos\omega'_M)^2}\;.
\end{aligned}
$$

C.1.4 Derivatives of $\int(1/\mathrm{d})\, d\ell d\ell'$

In this section we put all the formulas needed for the computation of the derivatives of $\int_{\mathbb{T}^2}\frac{1}{\mathrm{d}}\,d\ell\, d\ell'$.

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$$
\begin{aligned}
\frac{\partial}{\partial e}\int_{\mathbb{T}^2}\frac{1}{\mathrm{d}}\,d\ell d\ell' \;=\;& \frac{\partial}{\partial e}\left[\frac{1}{\sqrt{det(\mathcal{A})}}\right]\cdot\left\{\sum_{j=1}^4\int_{\theta_j}^{\theta_{j+1}}\sqrt{\left(d^+_{min}\right)^2+r_j^2(\theta)}\,d\theta-2\pi d^+_{min}\right\}+\\
& +\left[\frac{1}{\sqrt{det(\mathcal{A})}}\right]\cdot\left\{\frac{1}{2}\sum_{j=1}^4\int_{\theta_j}^{\theta_{j+1}}\frac{\frac{\partial}{\partial e}[\left(d^+_{min}\right)^2+r_j^2(\theta)]}{\sqrt{\left(d^+_{min}\right)^2+r_j^2(\theta)}}\,d\theta-2\pi\frac{\partial}{\partial e}d^+_{min}\right\}\;.
\end{aligned}
$$

The derivatives appearing in the previous formula are the following:

$$
\begin{aligned}
&\frac{\partial}{\partial e}\left[\frac{1}{\sqrt{det(\mathcal{A})}}\right]=-\frac{1}{2[det(\mathcal{A})]^{3/2}}\frac{\partial}{\partial e}\left(det(\mathcal{A})\right);\\
&\frac{\partial}{\partial e}\left[\left(d^+_{min}\right)^2\right]=2d^+_{nod}\frac{\partial}{\partial e}\left(d^+_{nod}\right)\left[\frac{\mathcal{G}^2\mathcal{G}'^2\sin^2 I_M}{det(\mathcal{A})}\right]+\left(d^+_{nod}\right)^2\frac{\mathcal{G}'^2\sin^2 I_M}{det(\mathcal{A})^2}\left[2det(\mathcal{A})\mathcal{G}\frac{\partial\mathcal{G}}{\partial e}-\mathcal{G}^2\frac{\partial}{\partial e}\left(det(\mathcal{A})\right)\right];\\
&\frac{\partial}{\partial e}(det(\mathcal{A}))=2\mathcal{G}'^2\mathcal{F}\frac{\partial\mathcal{F}}{\partial e}+2\mathcal{G}\frac{\partial\mathcal{G}}{\partial e}\left(\mathcal{F}'^2+\mathcal{G}'^2\sin^2 I_M\right)-2\mathcal{F}'\mathcal{G}'\cos I_M\left(\frac{\partial\mathcal{F}}{\partial e}\mathcal{G}+\mathcal{F}\frac{\partial\mathcal{G}}{\partial e}\right);
\end{aligned}
$$

$$\frac{\partial S_1}{\partial e} = \frac{\partial}{\partial e}\left[\frac{d^+_{nod}\mathcal{G}}{det(\mathcal{A})}\right]\left[\mathcal{F}\,\mathcal{G}'\cos I_M - \mathcal{F}'\mathcal{G}\right] + \left[\frac{d^+_{nod}\mathcal{G}}{det(\mathcal{A})}\right]\left[\frac{\partial \mathcal{F}}{\partial e}\mathcal{G}'\cos I_M - \mathcal{F}'\frac{\partial \mathcal{G}}{\partial e}\right];$$
$$\frac{\partial}{\partial e}\left[\frac{d^+_{nod}\mathcal{G}}{det(\mathcal{A})}\right] = \frac{1}{(det(\mathcal{A}))^2}\left\{det(\mathcal{A})\left[\frac{\partial}{\partial e}\left(d^+_{nod}\right)\mathcal{G} + d^+_{nod}\frac{\partial \mathcal{G}}{\partial e}\right] - d^+_{nod}\mathcal{G}\frac{\partial}{\partial e}\left(det(\mathcal{A})\right)\right\};$$
$$\frac{\partial S_2}{\partial e} = \frac{\partial}{\partial e}\left[\frac{d^+_{nod}\mathcal{G}'}{det(\mathcal{A})}\right]\left[\mathcal{F}\,\mathcal{G}' - \mathcal{F}'\mathcal{G}\cos I_M\right] + \left[\frac{d^+_{nod}\mathcal{G}'}{det(\mathcal{A})}\right]\left[\frac{\partial \mathcal{F}}{\partial e}\mathcal{G}' - \mathcal{F}'\frac{\partial \mathcal{G}}{\partial e}\cos I_M\right];$$
$$\frac{\partial}{\partial e}\left[\frac{d^+_{nod}\mathcal{G}'}{det(\mathcal{A})}\right] = \frac{\mathcal{G}'}{(det(\mathcal{A}))^2}\left[det(\mathcal{A})\frac{\partial}{\partial e}\left(d^+_{nod}\right) - d^+_{nod}\frac{det(\mathcal{A})}{\partial e}\right];$$
$$\frac{\partial r_1}{\partial e} = \frac{\tau}{(\rho\cos\theta - \sigma\sin\theta)^2}\left\{(\rho\cos\theta - \sigma\sin\theta)\left[\frac{\partial\rho}{\partial e}(\pi - \overline{\ell}' - S_1) - \rho\left(\frac{\partial\overline{\ell}'}{\partial e} + \frac{\partial S_1}{\partial e}\right)\right] - \right.$$
$$\left. -\rho(\pi - \overline{\ell}' - S_1)\left[\frac{\partial\rho}{\partial e}\cos\theta - \frac{\partial\sigma}{\partial e}\sin\theta\right]\right\};$$
$$\frac{\partial r_2}{\partial e} = \frac{1}{\sin\theta}\left\{\frac{\partial\rho}{\partial e}(\pi - \overline{\ell} - S_2) - \rho\left(\frac{\partial\overline{\ell}}{\partial e} + \frac{\partial S_2}{\partial e}\right)\right\};$$
$$\frac{\partial r_3}{\partial e} = -\frac{\tau}{(\rho\cos\theta - \sigma\sin\theta)^2}\left\{(\rho\cos\theta - \sigma\sin\theta)\left[\frac{\partial\rho}{\partial e}(\pi + \overline{\ell}' + S_1) + \rho\left(\frac{\partial\overline{\ell}'}{\partial e} + \frac{\partial S_1}{\partial e}\right)\right] - \right.$$
$$\left. -\rho(\pi + \overline{\ell}' + S_1)\left[\frac{\partial\rho}{\partial e}\cos\theta - \frac{\partial\sigma}{\partial e}\sin\theta\right]\right\};$$
$$\frac{\partial r_4}{\partial e} = -\frac{1}{\sin\theta}\left\{\frac{\partial\rho}{\partial e}(\pi + \overline{\ell} + S_2) + \rho\left(\frac{\partial\overline{\ell}}{\partial e} + \frac{\partial S_2}{\partial e}\right)\right\};$$
$$\frac{\partial\tau}{\partial e} = 0; \qquad \frac{\partial\sigma}{\partial e} = -\frac{1}{\tau}\left[\mathcal{F}'\frac{\partial \mathcal{F}}{\partial e} + \mathcal{G}'\cos I_M\frac{\partial \mathcal{G}}{\partial e}\right]; \qquad \frac{\partial\rho}{\partial e} = \frac{1}{2\tau\sqrt{det(\mathcal{A})}}\frac{\partial}{\partial e}\left(det(\mathcal{A})\right);$$
$$\frac{\partial\overline{\ell}}{\partial e} = -\frac{\partial k}{\partial e}; \qquad \frac{\partial\overline{\ell}'}{\partial e} = 0\,.$$

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$$\frac{\partial}{\partial\omega_M}\int_{\mathbb{T}^2}\frac{1}{\mathrm{d}}\,d\ell\,d\ell' = \frac{\partial}{\partial\omega_M}\left[\frac{1}{\sqrt{det(\mathcal{A})}}\right]\cdot\left\{\sum_{j=1}^{4}\int_{\theta_j}^{\theta_{j+1}}\sqrt{\left(d^+_{min}\right)^2 + r_j^2(\theta)}\,d\theta - 2\pi d^+_{min}\right\} +$$
$$+\left[\frac{1}{\sqrt{det(\mathcal{A})}}\right]\cdot\left\{\frac{1}{2}\sum_{j=1}^{4}\int_{\theta_j}^{\theta_{j+1}}\frac{\frac{\partial}{\partial\omega_M}[\left(d^+_{min}\right)^2 + r_j^2(\theta)]}{\sqrt{\left(d^+_{min}\right)^2 + r_j^2(\theta)}}\,d\theta - 2\pi\frac{\partial}{\partial\omega_M}d^+_{min}\right\}.$$

The derivatives appearing in the previous formula are the following:

$$\frac{\partial}{\partial\omega_M}\left[\frac{1}{\sqrt{det(\mathcal{A})}}\right] = -\frac{1}{2[det(\mathcal{A})]^{3/2}}\frac{\partial}{\partial e}\left(det(\mathcal{A})\right);$$

$$\frac{\partial}{\partial\omega_M}\left[(d^+_{min})^2\right] = 2d^+_{nod}\frac{\partial}{\partial\omega_M}\left(d^+_{nod}\right)\left[\frac{\mathcal{G}^2\mathcal{G}'^2\sin^2 I_M}{det(\mathcal{A})}\right]+$$
$$+(d^+_{nod})^2\frac{\mathcal{G}'^2\sin^2 I_M}{det(\mathcal{A})^2}\left[2det(\mathcal{A})\mathcal{G}\frac{\partial\mathcal{G}}{\partial\omega_M}-\mathcal{G}^2\frac{\partial}{\partial\omega_M}\left(det(\mathcal{A})\right)\right];$$

$$\frac{\partial}{\partial\omega_M}(det(\mathcal{A})) = 2\mathcal{G}'^2\mathcal{F}\frac{\partial\mathcal{F}}{\partial\omega_M}+2\mathcal{G}\frac{\partial\mathcal{G}}{\partial\omega_M}\left(\mathcal{F}'^2+\mathcal{G}'^2\sin^2 I_M\right)-2\mathcal{F}'\mathcal{G}'\cos I_M\left(\frac{\partial\mathcal{F}}{\partial\omega_M}\mathcal{G}+\mathcal{F}\frac{\partial\mathcal{G}}{\partial\omega_M}\right);$$

$$\frac{\partial S_1}{\partial\omega_M} = \frac{\partial}{\partial\omega_M}\left[\frac{d^+_{nod}\mathcal{G}}{det(\mathcal{A})}\right]\left[\mathcal{F}\mathcal{G}'\cos I_M-\mathcal{F}'\mathcal{G}\right]+\left[\frac{d^+_{nod}\mathcal{G}}{det(\mathcal{A})}\right]\left[\frac{\partial\mathcal{F}}{\partial\omega_M}\mathcal{G}'\cos I_M-\mathcal{F}'\frac{\partial\mathcal{G}}{\partial\omega_M}\right];$$

$$\frac{\partial}{\partial\omega_M}\left[\frac{d^+_{nod}\mathcal{G}}{det(\mathcal{A})}\right] = \frac{1}{(det(\mathcal{A}))^2}\left\{det(\mathcal{A})\left[\frac{\partial}{\partial\omega_M}\left(d^+_{nod}\right)\mathcal{G}+d^+_{nod}\frac{\partial\mathcal{G}}{\partial\omega_M}\right]-d^+_{nod}\mathcal{G}\frac{\partial}{\partial\omega_M}\left(det(\mathcal{A})\right)\right\};$$

$$\frac{\partial S_2}{\partial\omega_M} = \frac{\partial}{\partial\omega_M}\left[\frac{d^+_{nod}\mathcal{G}'}{det(\mathcal{A})}\right]\left[\mathcal{F}\mathcal{G}'-\mathcal{F}'\mathcal{G}\cos I_M\right]+\left[\frac{d^+_{nod}\mathcal{G}'}{det(\mathcal{A})}\right]\left[\frac{\partial\mathcal{F}}{\partial\omega_M}\mathcal{G}'-\mathcal{F}'\frac{\partial\mathcal{G}}{\partial\omega_M}\cos I_M\right];$$

$$\frac{\partial}{\partial\omega_M}\left[\frac{d^+_{nod}\mathcal{G}'}{det(\mathcal{A})}\right] = \frac{\mathcal{G}'}{(det(\mathcal{A}))^2}\left[det(\mathcal{A})\frac{\partial}{\partial\omega_M}\left(d^+_{nod}\right)-d^+_{nod}\frac{det(\mathcal{A})}{\partial\omega_M}\right];$$

$$\frac{\partial r_1}{\partial\omega_M} = \frac{\tau}{(\rho\cos\theta-\sigma\sin\theta)^2}\left\{(\rho\cos\theta-\sigma\sin\theta)\left[\frac{\partial\rho}{\partial\omega_M}(\pi-\overline{\ell}'-S_1)-\rho\left(\frac{\partial\overline{\ell}'}{\partial\omega_M}+\frac{\partial S_1}{\partial\omega_M}\right)\right]-\right.$$
$$\left.-\rho(\pi-\overline{\ell}'-S_1)\left[\frac{\partial\rho}{\partial\omega_M}\cos\theta-\frac{\partial\sigma}{\partial\omega_M}\sin\theta\right]\right\};$$

$$\frac{\partial r_2}{\partial\omega_M} = \frac{1}{\sin\theta}\left\{\frac{\partial\rho}{\partial\omega_M}(\pi-\overline{\ell}-S_2)-\rho\left(\frac{\partial\overline{\ell}}{\partial\omega_M}+\frac{\partial S_2}{\partial\omega_M}\right)\right\};$$

$$\frac{\partial r_3}{\partial\omega_M} = -\frac{\tau}{(\rho\cos\theta-\sigma\sin\theta)^2}\left\{(\rho\cos\theta-\sigma\sin\theta)\left[\frac{\partial\rho}{\partial\omega_M}(\pi+\overline{\ell}'+S_1)+\rho\left(\frac{\partial\overline{\ell}'}{\partial\omega_M}+\frac{\partial S_1}{\partial\omega_M}\right)\right]-\right.$$
$$\left.-\rho(\pi+\overline{\ell}'+S_1)\left[\frac{\partial\rho}{\partial\omega_M}\cos\theta-\frac{\partial\sigma}{\partial\omega_M}\sin\theta\right]\right\};$$

$$\frac{\partial r_4}{\partial\omega_M} = -\frac{1}{\sin\theta}\left\{\frac{\partial\rho}{\partial\omega_M}(\pi+\overline{\ell}+S_2)+\rho\left(\frac{\partial\overline{\ell}}{\partial\omega_M}+\frac{\partial S_2}{\partial\omega_M}\right)\right\};$$

$$\frac{\partial\tau}{\partial\omega_M}=0; \qquad \frac{\partial\sigma}{\partial\omega_M} = -\frac{1}{\tau}\left[\mathcal{F}'\frac{\partial\mathcal{F}}{\partial\omega_M}+\mathcal{G}'\cos I_M\frac{\partial\mathcal{G}}{\partial\omega_M}\right]; \qquad \frac{\partial\rho}{\partial\omega_M} = \frac{1}{2\tau\sqrt{det(\mathcal{A})}}\frac{\partial}{\partial\omega_M}\left(det(\mathcal{A})\right);$$

$$\frac{\partial\overline{\ell}}{\partial\omega_M} = -\frac{\partial k}{\partial e}; \qquad \frac{\partial\overline{\ell}'}{\partial\omega_M}=0\,.$$

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$$\frac{\partial}{\partial\omega'_M}\int_{\mathbb{T}^2}\frac{1}{\mathrm{d}}\,d\ell\,d\ell' \quad = \quad \frac{\partial}{\partial\omega'_M}\left[\frac{1}{\sqrt{det(\mathcal{A})}}\right]\cdot\left\{\sum_{j=1}^{4}\int_{\theta_j}^{\theta_{j+1}}\sqrt{\left(d^+_{min}\right)^2+r_j^2(\theta)}\,d\theta-2\pi d^+_{min}\right\}+$$

$$+\left[\frac{1}{\sqrt{det(\mathcal{A})}}\right]\cdot\left\{\frac{1}{2}\sum_{j=1}^{4}\int_{\theta_j}^{\theta_{j+1}}\frac{\frac{\partial}{\partial\omega'_M}[(d^+_{min})^2+r_j^2(\theta)]}{\sqrt{(d^+_{min})^2+r_j^2(\theta)}}d\theta-2\pi\frac{\partial}{\partial\omega'_M}d^+_{min}\right\}.$$

The derivatives appearing in the previous formula are the following:

$$\frac{\partial}{\partial\omega'_M}\left[\frac{1}{\sqrt{det(\mathcal{A})}}\right]=-\frac{1}{2[det(\mathcal{A})]^{3/2}}\frac{\partial}{\partial\omega'_M}\left(det(\mathcal{A})\right);$$

$$\frac{\partial}{\partial\omega'_M}\left[(d^+_{min})^2\right]=2d^+_{nod}\frac{\partial}{\partial\omega'_M}\left(d^+_{nod}\right)\left[\frac{\mathcal{G}^2\mathcal{G}'^2\sin^2 I_M}{det(\mathcal{A})}\right]+$$
$$+(d^+_{nod})^2\frac{\mathcal{G}^2\sin^2 I_M}{det(\mathcal{A})^2}\left[2det(\mathcal{A})\mathcal{G}'\frac{\partial\mathcal{G}'}{\partial\omega'_M}-\mathcal{G}'^2\frac{\partial}{\partial\omega'_M}\left(det(\mathcal{A})\right)\right];$$

$$\frac{\partial}{\partial\omega'_M}(det(\mathcal{A}))=2\mathcal{G}'\frac{\partial\mathcal{G}'}{\partial\omega'_M}\left(\mathcal{F}^2+\mathcal{G}^2\sin^2 I\right)+2\mathcal{F}'\frac{\partial\mathcal{F}'}{\partial\omega'_M}\mathcal{G}^2-2\mathcal{F}\,\mathcal{G}\cos I_M\left(\frac{\partial\mathcal{F}'}{\partial\omega'_M}\mathcal{G}'+\mathcal{F}'\frac{\partial\mathcal{G}'}{\partial\omega'_M}\right);$$

$$\frac{\partial S_1}{\partial\omega'_M}=\frac{\partial}{\partial\omega'_M}\left[\frac{d^+_{nod}\mathcal{G}}{det(\mathcal{A})}\right]\left[\mathcal{F}\,\mathcal{G}'\cos I_M-\mathcal{F}'\,\mathcal{G}\right]+\left[\frac{d^+_{nod}\mathcal{G}}{det(\mathcal{A})}\right]\left[\mathcal{F}\frac{\partial\mathcal{G}'}{\partial\omega'_M}\cos I_M-\mathcal{G}\frac{\partial\mathcal{F}'}{\partial\omega'_M}\right];$$
$$\frac{\partial}{\partial\omega'_M}\left[\frac{d^+_{nod}\mathcal{G}}{det(\mathcal{A})}\right]=\frac{\mathcal{G}}{(det(\mathcal{A}))^2}\left[det(\mathcal{A})\frac{\partial}{\partial\omega'_M}\left(d^+_{nod}\right)-d^+_{nod}\frac{det(\mathcal{A})}{\partial\omega'_M}\right];$$

$$\frac{\partial S_2}{\partial\omega'_M}=\frac{\partial}{\partial\omega'_M}\left[\frac{d^+_{nod}\mathcal{G}'}{det(\mathcal{A})}\right]\left[\mathcal{F}\,\mathcal{G}'-\mathcal{F}'\,\mathcal{G}\cos I_M\right]+\left[\frac{d^+_{nod}\mathcal{G}'}{det(\mathcal{A})}\right]\left[\mathcal{F}\frac{\partial\mathcal{G}'}{\partial\omega'_M}-\mathcal{G}\frac{\partial\mathcal{F}'}{\partial\omega'_M}\cos I_M\right];$$
$$\frac{\partial}{\partial\omega'_M}\left[\frac{d^+_{nod}\mathcal{G}'}{det(\mathcal{A})}\right]=\frac{1}{(det(\mathcal{A}))^2}\left\{det(\mathcal{A})\left[\frac{\partial}{\partial\omega'_M}\left(d^+_{nod}\right)\mathcal{G}'+d^+_{nod}\frac{\partial\mathcal{G}'}{\partial\omega'_M}\right]-d^+_{nod}\mathcal{G}'\frac{\partial}{\partial\omega'_M}\left(det(\mathcal{A})\right)\right\};$$

$$\frac{\partial r_1}{\partial\omega'_M}=\frac{\tau}{(\rho\cos\theta-\sigma\sin\theta)^2}\left\{(\rho\cos\theta-\sigma\sin\theta)\left[\frac{\partial\rho}{\partial\omega'_M}(\pi-\overline{\ell}'-S_1)-\rho\left(\frac{\partial\overline{\ell}'}{\partial\omega'_M}+\frac{\partial S_1}{\partial\omega'_M}\right)\right]-\right.$$
$$\left.-\rho(\pi-\overline{\ell}'-S_1)\left[\frac{\partial\rho}{\partial\omega'_M}\cos\theta-\frac{\partial\sigma}{\partial\omega'_M}\sin\theta\right]\right\};$$

$$\frac{\partial r_2}{\partial\omega'_M}=\frac{1}{\sin\theta}\left\{\frac{\partial\rho}{\partial\omega'_M}(\pi-\overline{\ell}-S_2)-\rho\left(\frac{\partial\overline{\ell}}{\partial\omega'_M}+\frac{\partial S_2}{\partial\omega'_M}\right)\right\};$$

$$\frac{\partial r_3}{\partial\omega'_M}=-\frac{\tau}{(\rho\cos\theta-\sigma\sin\theta)^2}\left\{(\rho\cos\theta-\sigma\sin\theta)\left[\frac{\partial\rho}{\partial\omega'_M}(\pi+\overline{\ell}'+S_1)+\rho\left(\frac{\partial\overline{\ell}'}{\partial\omega'_M}+\frac{\partial S_1}{\partial\omega'_M}\right)\right]-\right.$$
$$\left.-\rho(\pi+\overline{\ell}'+S_1)\left[\frac{\partial\rho}{\partial\omega'_M}\cos\theta-\frac{\partial\sigma}{\partial\omega'_M}\sin\theta\right]\right\};$$

$$\frac{\partial r_4}{\partial\omega'_M}=-\frac{1}{\sin\theta}\left\{\frac{\partial\rho}{\partial\omega'_M}(\pi+\overline{\ell}+S_2)+\rho\left(\frac{\partial\overline{\ell}}{\partial\omega'_M}+\frac{\partial S_2}{\partial\omega'_M}\right)\right\};$$

$$\frac{\partial\tau}{\partial\omega'_M}=\frac{1}{\tau}\left[\mathcal{F}'\frac{\partial\mathcal{F}'}{\partial\omega'_M}+\mathcal{G}'\frac{\partial\mathcal{G}'}{\partial\omega'_M}\right];$$

$$\frac{\partial\sigma}{\partial\omega'_M} = \frac{1}{\tau^2}\left\{\frac{\partial\tau}{\partial\omega'_M}[\mathcal{F}\mathcal{F}' + \mathcal{G}\mathcal{G}'\cos I_M] - \tau\left[\mathcal{F}\frac{\partial\mathcal{F}'}{\partial\omega'_M} + \mathcal{G}\cos I_M\frac{\partial\mathcal{G}'}{\partial\omega''_M}\right]\right\};$$
$$\frac{\partial\rho}{\partial\omega'_M} = -\frac{1}{\tau^2}\left[\frac{\partial\tau}{\partial\omega'_M}det(\mathcal{A}) - \frac{\tau}{2\sqrt{det(\mathcal{A})}}\frac{\partial}{\partial\omega'_M}(det(\mathcal{A}))\right];$$
$$\frac{\partial\overline{\ell}}{\partial\omega'_M} = 0; \qquad \frac{\partial\overline{\ell}'}{\partial\omega'_M} = -\frac{\partial k'}{\partial e}.$$

C.1.5 Derivatives of $\int(1/\mathrm{d}_1)\, d\ell d\ell'$

In this section we put all the formulas needed for the computation of the derivatives of the descending node crossing approximation $\int_{\mathbb{T}^2}\frac{1}{\mathrm{d}_1}\,d\ell d\ell'$.

DERIVATIVE WITH RESPECT TO e

$$\begin{aligned}\frac{\partial}{\partial e}\int_{\mathbb{T}^2}\frac{1}{\mathrm{d}_1}d\ell d\ell' \; &= \; \frac{\partial}{\partial e}\left[\frac{1}{\sqrt{det(\mathcal{A}_d)}}\right]\cdot\left\{\sum_{j=1}^{4}\int_{\theta_j^-}^{\theta_{j+1}^-}\sqrt{(d_{min}^-)^2 + (r_j^-)^2(\theta)}\,d\theta - 2\pi d_{min}^-\right\} + \\ &+\left[\frac{1}{\sqrt{det(\mathcal{A}_d)}}\right]\cdot\left\{\frac{1}{2}\sum_{j=1}^{4}\int_{\theta_j^-}^{\theta_{j+1}^-}\frac{\frac{\partial}{\partial e}[(d_{min}^-)^2 + (r_j^-)^2(\theta)]}{\sqrt{(d_{min}^-)^2 + (r_j^-)^2(\theta)}}\,d\theta - 2\pi\frac{\partial}{\partial e}d_{min}^-\right\}.\end{aligned}$$

where $r_j^-(\theta)$, with $j = 1\ldots4$, represent the lines r_j^- delimiting $\Xi_1[\overline{\mathbb{T}}^2]$ in polar coordinates:

$$r_1^-(\theta) = \frac{\rho_1\tau_1(\pi - S_1)}{\rho_1\cos\theta - \sigma_1\sin\theta}; \qquad r_2^-(\theta) = \frac{\rho_1(\pi - \overline{\ell}_1 - S_2)}{\sin\theta};$$

$$r_3^-(\theta) = \frac{-\rho_1\tau_1(\pi + S_1)}{\rho_1\cos\theta - \sigma_1\sin\theta}; \qquad r_4^-(\theta) = \frac{-\rho_1(\pi + \overline{\ell}_1 + S_2)}{\sin\theta};$$

while $\theta_1^- = \theta_5^- - 2\pi$ and θ_l^-, with $l = 2\ldots5$, are the angles defined by

$$0 < \theta_2^- < \theta_3^- < \pi < \theta_4^- < \theta_5^- < 2\pi;$$

$$\tan\theta_2^- = \frac{\rho_1(\pi - \overline{\ell}_1 - S_2)}{\sigma_1(\pi - \overline{\ell}_1 - S_2) + \tau_1(\pi - S_1)}; \quad \tan\theta_3^- = \frac{\rho_1(\pi - \overline{\ell}_1 - S_2)}{\sigma_1(\pi - \overline{\ell}_1 - S_2) - \tau_1(\pi + S_1)};$$
$$\tan\theta_4^- = \frac{\rho_1(\pi + \overline{\ell}_1 + S_2)}{\sigma_1(\pi + \overline{\ell}_1 + S_2) + \tau_1(\pi + S_1)}; \quad \tan\theta_5^- = \frac{\rho_1(\pi + \overline{\ell}_1 + S_2)}{\sigma_1(\pi + \overline{\ell}_1 + S_2) - \tau_1(\pi - S_1)}.$$

The derivatives appearing in the previous formula are the following:

$$\frac{\partial}{\partial e}\left[\frac{1}{\sqrt{det(\mathcal{A}_d)}}\right] = -\frac{1}{2[det(\mathcal{A}_d)]^{3/2}}\frac{\partial}{\partial e}(det(\mathcal{A}_d));$$
$$\frac{\partial}{\partial e}\left[(d_{min}^-)^2\right] = 2d_{nod}^-\frac{\partial}{\partial e}(d_{nod}^-)\left[\frac{\mathcal{G}_1^2\mathcal{G}_1'^2\sin^2 I_M}{det(\mathcal{A}_d)}\right] + (d_{nod}^-)^2\frac{\mathcal{G}_1'^2\sin^2 I_M}{det(\mathcal{A}_d)^2}\left[2det(\mathcal{A}_d)\mathcal{G}_1\frac{\partial\mathcal{G}_1}{\partial e} - \mathcal{G}_1^2\frac{\partial}{\partial e}(det(\mathcal{A}_d))\right];$$

$$\frac{\partial}{\partial e}(det(\mathcal{A}_d)) = 2\mathcal{G}_1'^2\mathcal{F}_1\frac{\partial \mathcal{F}_1}{\partial e} + 2\mathcal{G}_1\frac{\partial \mathcal{G}_1}{\partial e}\left(\mathcal{F}_1'^2 + \mathcal{G}_1'^2\sin^2 I_M\right) - 2\mathcal{F}_1'\mathcal{G}_1'\cos I_M\left(\frac{\partial \mathcal{F}_1}{\partial e}\mathcal{G}_1 + \mathcal{F}_1\frac{\partial \mathcal{G}_1}{\partial e}\right);$$

$$\frac{\partial S_1^-}{\partial e} = \frac{\partial}{\partial e}\left[\frac{d_{nod}^-\mathcal{G}_1}{det(\mathcal{A}_d)}\right]\left[\mathcal{F}_1'\mathcal{G}_1 - \mathcal{F}_1\mathcal{G}_1'\cos I_M\right] + \left[\frac{d_{nod}^-\mathcal{G}_1}{det(\mathcal{A}_d)}\right]\left[\mathcal{F}_1'\frac{\partial \mathcal{G}_1}{\partial e} - \frac{\partial \mathcal{F}_1}{\partial e}\mathcal{G}_1'\cos I_M\right];$$

$$\frac{\partial}{\partial e}\left[\frac{d_{nod}^-\mathcal{G}_1}{det(\mathcal{A}_d)}\right] = \frac{1}{(det(\mathcal{A}_d))^2}\left\{det(\mathcal{A}_d)\left[\frac{\partial}{\partial e}\left(d_{nod}^-\right)\mathcal{G}_1 + d_{nod}^-\frac{\partial \mathcal{G}_1}{\partial e}\right] - d_{nod}^-\mathcal{G}_1\frac{\partial}{\partial e}\left(det(\mathcal{A}_d)\right)\right\};$$

$$\frac{\partial S_2^-}{\partial e} = \frac{\partial}{\partial e}\left[\frac{d_{nod}^-\mathcal{G}_1'}{det(\mathcal{A}_d)}\right]\left[\mathcal{F}_1'\mathcal{G}_1\cos I_M - \mathcal{F}_1\mathcal{G}_1'\right] + \left[\frac{d_{nod}^-\mathcal{G}_1'}{det(\mathcal{A}_d)}\right]\left[\mathcal{F}_1'\frac{\partial \mathcal{G}_1}{\partial e}\cos I_M - \frac{\partial \mathcal{F}_1}{\partial e}\mathcal{G}_1'\right];$$

$$\frac{\partial}{\partial e}\left[\frac{d_{nod}^-\mathcal{G}_1'}{det(\mathcal{A}_d)}\right] = \frac{\mathcal{G}_1'}{(det(\mathcal{A}_d))^2}\left[det(\mathcal{A}_d)\frac{\partial}{\partial e}\left(d_{nod}^-\right) - d_{nod}^-\frac{det(\mathcal{A}_d)}{\partial e}\right];$$

$$\frac{\partial r_1^-}{\partial e} = \frac{\tau_1}{(\rho_1\cos\theta - \sigma_1\sin\theta)^2}\left\{(\rho_1\cos\theta - \sigma_1\sin\theta)\left[\frac{\partial \rho_1}{\partial e}(\pi - \overline{\ell}_1' - S_1^-) - \rho_1\left(\frac{\partial \overline{\ell}_1'}{\partial e} + \frac{\partial S_1^-}{\partial e}\right)\right] - \right.$$
$$\left. -\rho_1(\pi - \overline{\ell}_1' - S_1^-)\left[\frac{\partial \rho_1}{\partial e}\cos\theta - \frac{\partial \sigma_1}{\partial e}\sin\theta\right]\right\};$$

$$\frac{\partial r_2^-}{\partial e} = \frac{1}{\sin\theta}\left\{\frac{\partial \rho_1}{\partial e}(\pi - \overline{\ell}_1 - S_2^-) - \rho_1\left(\frac{\partial \overline{\ell}_1}{\partial e} + \frac{\partial S_2^-}{\partial e}\right)\right\};$$

$$\frac{\partial r_3^-}{\partial e} = -\frac{\tau_1}{(\rho_1\cos\theta - \sigma_1\sin\theta)^2}\left\{(\rho_1\cos\theta - \sigma_1\sin\theta)\left[\frac{\partial \rho_1}{\partial e}(\pi + \overline{\ell}_1' + S_1^-) + \rho_1\left(\frac{\partial \overline{\ell}_1'}{\partial e} + \frac{\partial S_1^-}{\partial e}\right)\right] - \right.$$
$$\left. -\rho_1(\pi + \overline{\ell}_1' + S_1^-)\left[\frac{\partial \rho_1}{\partial e}\cos\theta - \frac{\partial \sigma_1}{\partial e}\sin\theta\right]\right\};$$

$$\frac{\partial r_4^-}{\partial e} = -\frac{1}{\sin\theta}\left\{\frac{\partial \rho_1}{\partial e}(\pi + \overline{\ell}_1 + S_2^-) + \rho_1\left(\frac{\partial \overline{\ell}_1}{\partial e} + \frac{\partial S_2^-}{\partial e}\right)\right\};$$

$$\frac{\partial \tau_1}{\partial e} = 0; \qquad \frac{\partial \sigma_1}{\partial e} = -\frac{1}{\tau_1}\left[\mathcal{F}_1'\frac{\partial \mathcal{F}_1}{\partial e} + \mathcal{G}_1'\cos I_M\frac{\partial \mathcal{G}_1}{\partial e}\right]; \qquad \frac{\partial \rho_1}{\partial e} = \frac{1}{2\tau_1\sqrt{det(\mathcal{A}_d)}}\frac{\partial}{\partial e}\left(det(\mathcal{A}_d)\right);$$

$$\frac{\partial \overline{\ell}_1}{\partial e} = -\frac{\partial k_1}{\partial e}; \qquad \frac{\partial \overline{\ell}_1'}{\partial e} = 0\,.$$

DERIVATIVE WITH RESPECT TO ω_M

$$\frac{\partial}{\partial \omega_M}\int_{\mathbb{T}^2}\frac{1}{\mathrm{d}_1}\,d\ell\,d\ell' = \frac{\partial}{\partial \omega_M}\left[\frac{1}{\sqrt{det(\mathcal{A}_d)}}\right]\cdot\left\{\sum_{j=1}^{4}\int_{\theta_j^-}^{\theta_{j+1}^-}\sqrt{\left(d_{min}^-\right)^2 + \left(r_j^-\right)^2(\theta)}\,d\theta - 2\pi d_{min}^-\right\} +$$
$$+\left[\frac{1}{\sqrt{det(\mathcal{A}_d)}}\right]\cdot\left\{\frac{1}{2}\sum_{j=1}^{4}\int_{\theta_j^-}^{\theta_{j+1}^-}\frac{\frac{\partial}{\partial \omega_M}\left[\left(d_{min}^-\right)^2 + \left(r_j^-\right)^2(\theta)\right]}{\sqrt{\left(d_{min}^-\right)^2 + \left(r_j^-\right)^2(\theta)}}\,d\theta - 2\pi\frac{\partial}{\partial \omega_M}d_{min}^-\right\}.$$

The derivatives appearing in the previous formula are the following:

$$\frac{\partial}{\partial\omega_M}\left[\frac{1}{\sqrt{det(\mathcal{A})}}\right]=-\frac{1}{2[det(\mathcal{A})]^{3/2}}\frac{\partial}{\partial e}\left(det(\mathcal{A}_d)\right);$$

$$\frac{\partial}{\partial\omega_M}\left[(d_{min}^{-})^2\right]=2d_{nod}^{-}\frac{\partial}{\partial\omega_M}\left(d_{nod}^{-}\right)\left[\frac{\mathcal{G}_1^2\mathcal{G}_1'^2\sin^2 I_M}{det(\mathcal{A}_d)}\right]+$$
$$+(d_{nod}^{-})^2\frac{\mathcal{G}_1'^2\sin^2 I_M}{det(\mathcal{A}_d)^2}\left[2det(\mathcal{A}_d)\mathcal{G}_1\frac{\partial\mathcal{G}_1}{\partial\omega_M}-\mathcal{G}_1^2\frac{\partial}{\partial\omega_M}\left(det(\mathcal{A}_d)\right)\right];$$

$$\frac{\partial}{\partial\omega_M}(det(\mathcal{A}_d))=2\mathcal{G}_1'^2\mathcal{F}_1\frac{\partial\mathcal{F}_1}{\partial\omega_M}+2\mathcal{G}_1\frac{\partial\mathcal{G}_1}{\partial\omega_M}\left(\mathcal{F}_1'^2+\mathcal{G}_1'^2\sin^2 I_M\right)-2\mathcal{F}_1'\mathcal{G}_1'\cos I_M\left(\frac{\partial\mathcal{F}_1}{\partial\omega_M}\mathcal{G}_1+\mathcal{F}_1\frac{\partial\mathcal{G}_1}{\partial\omega_M}\right);$$

$$\frac{\partial S_1^-}{\partial\omega_M}=\frac{\partial}{\partial\omega_M}\left[\frac{d_{nod}^-\mathcal{G}_1}{det(\mathcal{A}_d)}\right]\left[\mathcal{F}_1'\mathcal{G}_1-\mathcal{F}_1\mathcal{G}_1'\cos I_M\right]+\left[\frac{d_{nod}^-\mathcal{G}_1}{det(\mathcal{A}_d)}\right]\left[\mathcal{F}_1'\frac{\partial\mathcal{G}_1}{\partial\omega_M}-\frac{\partial\mathcal{F}_1}{\partial\omega_M}\mathcal{G}_1'\cos I_M\right];$$

$$\frac{\partial}{\partial\omega_M}\left[\frac{d_{nod}^-\mathcal{G}_1}{det(\mathcal{A}_d)}\right]=\frac{1}{(det(\mathcal{A}_d))^2}\left\{det(\mathcal{A}_d)\left[\frac{\partial}{\partial\omega_M}\left(d_{nod}^-\right)\mathcal{G}_1+d_{nod}^-\frac{\partial\mathcal{G}_1}{\partial\omega_M}\right]-d_{nod}^-\mathcal{G}_1\frac{\partial}{\partial\omega_M}\left(det(\mathcal{A}_d)\right)\right\};$$

$$\frac{\partial S_2^-}{\partial\omega_M}=\frac{\partial}{\partial\omega_M}\left[\frac{d_{nod}^-\mathcal{G}_1'}{det(\mathcal{A}_d)}\right]\left[\mathcal{F}_1'\mathcal{G}_1\cos I_M-\mathcal{F}_1\mathcal{G}_1'\right]+\left[\frac{d_{nod}^-\mathcal{G}_1'}{det(\mathcal{A}_d)}\right]\left[\mathcal{F}_1'\frac{\partial\mathcal{G}_1}{\partial\omega_M}\cos I_M-\frac{\partial\mathcal{F}_1}{\partial\omega_M}\mathcal{G}_1'\right];$$

$$\frac{\partial}{\partial\omega_M}\left[\frac{d_{nod}^-\mathcal{G}_1'}{det(\mathcal{A}_d)}\right]=\frac{\mathcal{G}_1'}{(det(\mathcal{A}_d))^2}\left[det(\mathcal{A}_d)\frac{\partial}{\partial\omega_M}\left(d_{nod}^-\right)-d_{nod}^-\frac{det(\mathcal{A}_d)}{\partial\omega_M}\right];$$

$$\frac{\partial r_1^-}{\partial\omega_M}=\frac{\tau_1}{(\rho_1\cos\theta-\sigma_1\sin\theta)^2}\left\{(\rho_1\cos\theta-\sigma_1\sin\theta)\left[\frac{\partial\rho_1}{\partial\omega_M}(\pi-\overline{\ell}_1'-S_1^-)-\rho_1\left(\frac{\partial\overline{\ell}_1'}{\partial\omega_M}+\frac{\partial S_1^-}{\partial\omega_M}\right)\right]-\right.$$
$$\left.-\rho_1(\pi-\overline{\ell}_1'-S_1^-)\left[\frac{\partial\rho_1}{\partial\omega_M}\cos\theta-\frac{\partial\sigma_1}{\partial\omega_M}\sin\theta\right]\right\};$$

$$\frac{\partial r_2^-}{\partial\omega_M}=\frac{1}{\sin\theta}\left\{\frac{\partial\rho_1}{\partial\omega_M}(\pi-\overline{\ell}_1-S_2^-)-\rho_1\left(\frac{\partial\overline{\ell}_1}{\partial\omega_M}+\frac{\partial S_2^-}{\partial\omega_M}\right)\right\};$$

$$\frac{\partial r_3^-}{\partial\omega_M}=-\frac{\tau_1}{(\rho_1\cos\theta-\sigma_1\sin\theta)^2}\left\{(\rho_1\cos\theta-\sigma_1\sin\theta)\left[\frac{\partial\rho_1}{\partial\omega_M}(\pi+\overline{\ell}_1'+S_1^-)+\rho_1\left(\frac{\partial\overline{\ell}_1'}{\partial\omega_M}+\frac{\partial S_1^-}{\partial\omega_M}\right)\right]-\right.$$
$$\left.-\rho_1(\pi+\overline{\ell}_1'+S_1^-)\left[\frac{\partial\rho_1}{\partial\omega_M}\cos\theta-\frac{\partial\sigma_1}{\partial\omega_M}\sin\theta\right]\right\};$$

$$\frac{\partial r_4^-}{\partial\omega_M}=-\frac{1}{\sin\theta}\left\{\frac{\partial\rho_1}{\partial\omega_M}(\pi+\overline{\ell}_1+S_2^-)+\rho_1\left(\frac{\partial\overline{\ell}_1}{\partial\omega_M}+\frac{\partial S_2^-}{\partial\omega_M}\right)\right\};$$

$$\frac{\partial\tau_1}{\partial\omega_M}=0;\qquad\frac{\partial\sigma_1}{\partial\omega_M}=-\frac{1}{\tau_1}\left[\mathcal{F}_1'\frac{\partial\mathcal{F}_1}{\partial\omega_M}+\mathcal{G}_1'\cos I_M\frac{\partial\mathcal{G}_1}{\partial\omega_M}\right];\qquad\frac{\partial\rho_1}{\partial\omega_M}=\frac{1}{2\tau_1\sqrt{det(\mathcal{A}_d)}}\frac{\partial}{\partial\omega_M}\left(det(\mathcal{A}_d)\right);$$

$$\frac{\partial\overline{\ell}_1}{\partial\omega_M}=-\frac{\partial k_1}{\partial e};\qquad\frac{\partial\overline{\ell}_1'}{\partial\omega_M}=0\,.$$

DERIVATIVE WITH RESPECT TO ω_M'

$$\frac{\partial}{\partial \omega'_M}\int_{\mathbb{T}^2}\frac{1}{\mathrm{d}_1}\,d\ell\, d\ell' \quad = \quad \frac{\partial}{\partial \omega'_M}\left[\frac{1}{\sqrt{det(\mathcal{A}_d)}}\right]\cdot\left\{\sum_{j=1}^{4}\int_{\theta_j^-}^{\theta_{j+1}^-}\sqrt{(d_{min}^-)^2+(r_j^-)^2(\theta)}\,d\theta-2\pi d_{min}^-\right\}+$$

$$+\left[\frac{1}{\sqrt{det(\mathcal{A}_d)}}\right]\cdot\left\{\frac{1}{2}\sum_{j=1}^{4}\int_{\theta_j^-}^{\theta_{j+1}^-}\frac{\frac{\partial}{\partial\omega'_M}[(d_{min}^-)^2+(r_j^-)^2(\theta)]}{\sqrt{(d_{min}^-)^2+(r_j^-)^2(\theta)}}\,d\theta-2\pi\frac{\partial}{\partial\omega'_M}d_{min}^-\right\}\;.$$

The derivatives appearing in the previous formula are the following:

$$\frac{\partial}{\partial\omega'_M}\left[\frac{1}{\sqrt{det(\mathcal{A}_d)}}\right]=-\frac{1}{2[det(\mathcal{A}_d)]^{3/2}}\frac{\partial}{\partial\omega'_M}\left(det(\mathcal{A}_d)\right);$$

$$\frac{\partial}{\partial\omega'_M}\left[(d_{min}^-)^2\right]=2d_{nod}^-\frac{\partial}{\partial\omega'_M}\left(d_{nod}^-\right)\left[\frac{\mathcal{G}_1^2\mathcal{G}_1'^2\sin^2 I_M}{det(\mathcal{A}_d)}\right]+$$

$$+(d_{nod}^-)^2\frac{\mathcal{G}_1^2\sin^2 I_M}{det(\mathcal{A}_d)^2}\left[2det(\mathcal{A}_d)\,\mathcal{G}_1'\frac{\partial\mathcal{G}_1'}{\partial\omega'_M}-\mathcal{G}_1'^2\frac{\partial}{\partial\omega'_M}\left(det(\mathcal{A}_d)\right)\right];$$

$$\frac{\partial}{\partial\omega'_M}(det(\mathcal{A}_d))=2\mathcal{G}_1'\frac{\partial\mathcal{G}_1'}{\partial\omega'_M}\left(\mathcal{F}_1^2+\mathcal{G}_1^2\sin^2 I\right)+2\mathcal{F}_1'\frac{\partial\mathcal{F}_1'}{\partial\omega'_M}\mathcal{G}_1^2-2\mathcal{F}_1\mathcal{G}_1\cos I_M\left(\frac{\partial\mathcal{F}_1'}{\partial\omega'_M}\mathcal{G}_1'+\mathcal{F}_1'\frac{\partial\mathcal{G}_1'}{\partial\omega'_M}\right);$$

$$\frac{\partial S_1^-}{\partial\omega'_M}=\frac{\partial}{\partial\omega'_M}\left[\frac{d_{nod}^-\mathcal{G}_1}{det(\mathcal{A}_d)}\right]\left[\mathcal{F}_1'\mathcal{G}_1-\mathcal{F}_1\mathcal{G}_1'\cos I_M\right]+\left[\frac{d_{nod}^-\mathcal{G}_1}{det(\mathcal{A}_d)}\right]\left[\mathcal{G}_1\frac{\partial\mathcal{F}_1'}{\partial\omega'_M}-\mathcal{F}_1\frac{\partial\mathcal{G}_1'}{\partial\omega'_M}\cos I_M\right];$$

$$\frac{\partial}{\partial\omega'_M}\left[\frac{d_{nod}^-\mathcal{G}_1}{det(\mathcal{A}_d)}\right]=\frac{\mathcal{G}_1}{(det(\mathcal{A}_d))^2}\left[det(\mathcal{A}_d)\frac{\partial}{\partial\omega'_M}\left(d_{nod}^-\right)-d_{nod}^-\frac{det(\mathcal{A}_d)}{\partial\omega'_M}\right];$$

$$\frac{\partial S_2^-}{\partial\omega'_M}=\frac{\partial}{\partial\omega'_M}\left[\frac{d_{nod}^-\mathcal{G}_1'}{det(\mathcal{A}_d)}\right]\left[\mathcal{F}_1'\mathcal{G}_1\cos I_M-\mathcal{F}_1\mathcal{G}_1'\right]+\left[\frac{d_{nod}^-\mathcal{G}_1'}{det(\mathcal{A}_d)}\right]\left[\mathcal{G}_1\frac{\partial\mathcal{F}_1'}{\partial\omega'_M}\cos I_M-\mathcal{F}_1\frac{\partial\mathcal{G}_1'}{\partial\omega'_M}\right];$$

$$\frac{\partial}{\partial\omega'_M}\left[\frac{d_{nod}^-\mathcal{G}_1'}{det(\mathcal{A}_d)}\right]=\frac{1}{(det(\mathcal{A}_d))^2}\left\{det(\mathcal{A}_d)\left[\frac{\partial}{\partial\omega'_M}\left(d_{nod}^-\right)\mathcal{G}_1'+d_{nod}^-\frac{\partial\mathcal{G}_1'}{\partial\omega'_M}\right]-d_{nod}^-\mathcal{G}_1'\frac{\partial}{\partial\omega'_M}\left(det(\mathcal{A}_d)\right)\right\};$$

$$\frac{\partial r_1^-}{\partial\omega'_M}=\frac{\tau_1}{(\rho_1\cos\theta-\sigma_1\sin\theta)^2}\left\{(\rho_1\cos\theta-\sigma_1\sin\theta)\left[\frac{\partial\rho_1}{\partial\omega'_M}(\pi-\overline{\ell}_1'-S_1^-)-\rho_1\left(\frac{\partial\overline{\ell}_1'}{\partial\omega'_M}+\frac{\partial S_1^-}{\partial\omega'_M}\right)\right]-\right.$$

$$\left.-\rho_1(\pi-\overline{\ell}_1'-S_1^-)\left[\frac{\partial\rho_1}{\partial\omega'_M}\cos\theta-\frac{\partial\sigma_1}{\partial\omega'_M}\sin\theta\right]\right\};$$

$$\frac{\partial r_2^-}{\partial\omega'_M}=\frac{1}{\sin\theta}\left\{\frac{\partial\rho_1}{\partial\omega'_M}(\pi-\overline{\ell}_1-S_2^-)-\rho_1\left(\frac{\partial\overline{\ell}_1}{\partial\omega'_M}+\frac{\partial S_2^-}{\partial\omega'_M}\right)\right\};$$

$$\frac{\partial r_3^-}{\partial\omega'_M}=-\frac{\tau_1}{(\rho_1\cos\theta-\sigma_1\sin\theta)^2}\left\{(\rho_1\cos\theta-\sigma_1\sin\theta)\left[\frac{\partial\rho_1}{\partial\omega'_M}(\pi+\overline{\ell}_1'+S_1^-)+\rho_1\left(\frac{\partial\overline{\ell}_1'}{\partial\omega'_M}+\frac{\partial S_1^-}{\partial\omega'_M}\right)\right]-\right.$$

$$\left.-\rho_1(\pi+\overline{\ell}_1'+S_1^-)\left[\frac{\partial\rho_1}{\partial\omega'_M}\cos\theta-\frac{\partial\sigma_1}{\partial\omega'_M}\sin\theta\right]\right\};$$

$$\frac{\partial r_4^-}{\partial\omega'_M}=-\frac{1}{\sin\theta}\left\{\frac{\partial\rho_1}{\partial\omega'_M}(\pi+\overline{\ell}_1+S_2^-)+\rho_1\left(\frac{\partial\overline{\ell}_1}{\partial\omega'_M}+\frac{\partial S_2^-}{\partial\omega'_M}\right)\right\};$$

$$
\begin{aligned}
\frac{\partial \tau_1}{\partial \omega'_M} &= \frac{1}{\tau_1}\left[\mathcal{F}_1'\frac{\partial \mathcal{F}_1'}{\partial \omega'_M} + \mathcal{G}_1'\frac{\partial \mathcal{G}_1'}{\partial \omega'_M}\right]\,;\\
\frac{\partial \sigma_1}{\partial \omega'_M} &= \frac{1}{\tau_1^2}\left\{\frac{\partial \tau_1}{\partial \omega'_M}[\mathcal{F}_1\mathcal{F}_1' + \mathcal{G}_1\mathcal{G}_1'\cos I_M] - \tau_1\left[\mathcal{F}_1\frac{\partial \mathcal{F}_1'}{\partial \omega'_M} + \mathcal{G}_1\cos I_M\frac{\partial \mathcal{G}_1'}{\partial \omega''_M}\right]\right\}\,;\\
\frac{\partial \rho_1}{\partial \omega'_M} &= -\frac{1}{\tau_1^2}\left[\frac{\partial \tau_1}{\partial \omega'_M}det(\mathcal{A}_d) - \frac{\tau_1}{2\sqrt{det(\mathcal{A}_d)}}\frac{\partial}{\partial \omega'_M}\left(det(\mathcal{A}_d)\right)\right]\,;\\
\frac{\partial \overline{\ell}_1}{\partial \omega'_M} &= 0\,; \qquad\qquad \frac{\partial \overline{\ell}_1'}{\partial \omega'_M} = -\frac{\partial k_1'}{\partial e}\,.
\end{aligned}
$$

Appendix D

D.1 Algebraic Geometry results

In this section we give the definition and the basic ideas of *subresultant theory*. For more details see (see [24]).

D.1.1 Subresultant of two polynomials

We use the same notations of Chapter 10 and give the definitions and the results in the case of our interest:

DEFINITION 28. *Given the Sylvester matrix S_0, we call principal subresultant coefficients of order 1 and 2 ($PSRC_1(p,q)$ and $PSRC_2(p,q)$ respectively) of $p(t,s)$ and $q(t,s)$, with respect to s, the determinant of the matrixes*

$$S_1 = \begin{pmatrix} \alpha & 0 & 0 & B \\ \beta & \alpha & 0 & 0 \\ \gamma & \beta & \alpha & D \\ 0 & \gamma & \beta & -A \end{pmatrix}$$

and

$$S_2 = \begin{pmatrix} \alpha & 0 \\ \beta & \alpha \end{pmatrix}$$

REMARK 38. The first matrix is obtained from S_0 deleting the first and the last row and the first and the last column. The second one is obtained by performing the same operation on S_1.

If we denote with $PSRC_0(p,q)$ the classical resultant, we have

THEOREM 24. *The Greatest Common Divisor of $p(t,s)$ and $q(t,s)$ has degree ≥ 2 in s if and only if $PSRC_0(p,q) = PSRC_1(p,q) = 0$ and it has degree ≥ 3 if and only if $PSRC_0(p,q) = PSRC_1(p,q) = PSRC_2(p,q) = 0$.*

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