Proper elements for Earth crossing asteroids

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Abstract

Proper elements for 987 known Near Earth Asteroids (NEA) have been computed using a generalized averaging principle, valid also for planet crossing orbits. This allows to assess the maximum and minimum eccentricity and inclination resulting from secular perturbations, to determine which planetary orbits can be crossed, and to find the encounter conditions, which can also be used to search for parent bodies of meteor streams. Proper frequencies for these asteroids have also been computed; then it has been possible to draw a map of the NEA's affected by the main secular resonances. This paper describes the methods used in this computation, discusses the main qualitative properties of the results, and announces the availability of the proper elements online.

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1 Introduction

In the study of Dynamics a very important role is played by the integrals of the motion, that is by quantities that are constant during the time evolution of a dynamical system. When the dynamics is non integrable, as it is the case for the N-body problem, such quantities do not exist. Then it is very useful to compute quasi integrals of the motion: they are quantities changing very slowly, and can be considered approximately constant over time intervals not too long. In the context of asteroid dynamics, the quasi integrals are called proper elements.

Proper elements have been computed analytically for the main belt asteroids [Milani and Knezevič 1992, Milani and Knezevič 1994] and have been successfully used to determine asteroid collisional families and to understand the dynamical structure of the asteroid belt, in particular the relevance of secular resonances. This has been possible because these proper elements have been proven to be stable over time scales of the order of 10^7 years. Over an even longer time scale, in many cases slow diffusion processes change them by significant amounts [Milani and Farinella 1994], nevertheless the stability of the proper elements is enough to identify a large number of reliable collisional families [Zappalà et al. 1994].

Another method used to compute proper elements [Williams 1969] is the semianalytic one, that is based upon the averaging, by quadrature, of the equations of motion over the fast angle variables, the ones with periods of a few years. Semianalytic proper elements have been used effectively for asteroids belonging to the main belt but with inclination and eccentricity too high for accurate analytical theories [Lemaitre and Morbidelli 1994].

A small fraction of the known asteroids, and a significant fraction of the comets, are on orbits which can cross the orbits of the terrestrial planets and/or the giant planets. The Earth-crossing population is especially important for the role it plays in the formation of craters on the Earth and on the Moon. It would be desirable to be capable to compute proper elements also for this class of orbits, for the following reasons. First, to detect the possibility and compute the probability of collision of such objects with the Earth over intermediate time scales (thousands to tens of thousands years). Second, to identify the objects whose long term orbital evolution is controlled by one or more of the main secular resonances, to draw a map of the dynamical structure of the Earth crossing region in phase space. Third, to identify meteor streams (that can be observed only when they are crossing the orbit of the Earth) and to give a criterion to be employed in the identification of their parent bodies.

The very fact that the orbits in this class can cross the orbit of some planets results in the presence of the mathematical singularity of collision, either along the orbit or nearby in the phase space. This singularity results in both strong divergence of the perturbative series used in the analytical theories and in the divergence of the quadratures used in the semianalytic averaging. Thus both techniques used to compute proper elements for main belt asteroids are inapplicable to Earth-crossing orbits.

On the other hand we have to realize that the goals for which we would like to have proper elements are very different in this case, and the required stability times are much shorter: for most applications, time spans between 1 000 and 100 000 years are long enough. Over longer time spans the dynamics is dominated by large changes in orbital elements, including semimajor axis, resulting from close approaches and the effects of secular resonances. This kind of dynamics is strongly chaotic to the point of being impossible to describe other than in a statistical sense. Thus even a rougher approximation could be used, provided it is shown to be mathematically consistent and numerically computable, if it can provide information stable over time spans comparable to one cycle of the slow, secular arguments, such as the argument of perihelion ω.

In the paper [Gronchi and Milani 1998] we have developed the mathematical theory of such an approximation, applicable even to orbits which undergo crossings with the orbit of a major planet. In this paper we improve upon this theory, and we derive from it a computationally realistic and numerically stable algorithm to compute the right hand side of the averaged equations of motion (Section 2). We have based a numerical integration method for the averaged equations (Section 3) on this mathematical foundation. In this way we have computed proper elements for all the Near-Earth Asteroids (NEA), and made them available in a catalog (as announced in Section 4). These proper elements allow to discuss qualitative properties of the near-Earth orbits, such as the occurrence and frequency of node crossings with our planet (Section 5), and the occurrence of secular resonances (Section 6). We conclude (in Section 7) by stressing that this is just the first step in the procedure to manufacture reliable and accurate proper element catalogs for NEA (and in the future also for comets and meteors). This needs to be followed by a long series of tests, in which stability problems will be identified, and successive improvements of the computational algorithm will be implemented, as it happened for the analytical proper elements of main belt asteroids [Milani and Knezevič 1988, Milani and Knezevič 1998].

2 Averaging theory for planet crossing orbits

The model employed in this theory is a N-body problem restricted and circular, that is the asteroid is considered as a massless object and the Solar system major planets are on circular and coplanar orbits. Let us describe, for simplicity, the averaging theory for a 3 body problem: Sun-Earth-asteroid.

The Hamiltonian of the system has the form $H = H_0 - R$, where H_0 is the 2-body Hamiltonian and R is the perturbing function

$$R = k^2 \mu \left[\frac{1}{|\vec{x} - \vec{x}'|} - \frac{(\vec{x} \cdot \vec{x}')}{x'^3} \right]$$

(k is Gauss' gravitational constant, $\mu = m_\oplus / m_\odot$ is the ratio of the mass of the Earth and the mass of the Sun, $\vec{x}$ and $\vec{x}'$ are the positions of the asteroid and the Earth in a Heliocentric reference frame). In the following we shall exploit the well known fact that R can also be written as a function of Delaunay variables (ℓ, g, z, L, G, Z) and $(\ell', g', z', L', G', Z')$, of the asteroid and of the Earth, respectively.

If the two orbits intersect, then the perturbing function has a first order polar singularity given by the direct perturbation term $1/|\vec{x} - \vec{x}'|$; however the integral average of the perturbing function over the fast angle variables ℓ, ℓ'

$$\overline{R} = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} R \, d\ell d\ell' \tag{1}$$

is a convergent improper integral.

By a classical result the integral average of the indirect perturbation term

$$k^2 \mu \, \frac{(\vec{x} \cdot \vec{x}')}{x'^3}$$

is zero; therefore only the direct perturbation plays a role in this theory.

2.1 Semianalytic theories

The secular evolution of the main belt asteroids, whose orbits are not planet crossing (so that the Hamiltonian has no singularity), have been studied by applying the averaging principle [Arnold 1976]: when there is *no mean motion resonance* between the fast angle variables, the long term evolution of the asteroids can be computed to first order in μ by averaging the right-hand side of Hamilton equations over the fast angles (in this case the mean anomaly ℓ and the phase ℓ' of the circular orbit of the planet). So the equations to be solved are:

$$\begin{cases} \dot{g} = -\overline{\frac{\partial R}{\partial G}} \\ \dot{z} = -\overline{\frac{\partial R}{\partial Z}} \end{cases} \qquad\qquad \begin{cases} \dot{G} = \overline{\frac{\partial R}{\partial g}} \\ \dot{Z} = \overline{\frac{\partial R}{\partial z}} = 0 \end{cases} \tag{2}$$

where the *overline bar* means an integral average as in equation (1). Z, the component of the angular momentum orthogonal to the plane of the planetary orbits, and L, hence the mean semimajor axis a, are integrals for the averaged system because ℓ and z (which coincides with Ω, the longitude of the node) are cyclic variables.

This principle has been used to compute proper elements even beyond the circular planetary orbits approximation, by setting up a suitable perturbation theory [Williams 1969]; in more recent theories, even a significant part of the effects of second order in μ have been taken into account [Lemaitre and Morbidelli 1994]. However, this has not been possible so far for planet crossing orbits.

2.2 Semianalytic theories with collisions

When the target is to study the secular evolution of a planet crossing object, the first order polar singularity of the perturbing function R at the collision points in the phase space results in a second order polar singularity in its derivatives. Then the integral average of these derivatives, indicated in equations (2), is divergent, thus the classical averaging principle does not make sense.

However, at all the points in the phase space where the asteroid orbit is not planet crossing the derivatives of the perturbing function are regular functions, without singularities; there, by the *theorem of differentiation under the integral sign* (see [Fleming 1964]), equations (2) are equivalent to the following

$$\begin{cases} \dot{g} = -\frac{\partial \overline{R}}{\partial G} \\ \dot{z} = -\frac{\partial \overline{R}}{\partial Z} \end{cases} \qquad\qquad \begin{cases} \dot{G} = \frac{\partial \overline{R}}{\partial g} \\ \dot{Z} = \frac{\partial \overline{R}}{\partial z} = 0 \end{cases} \tag{3}$$

where the integral average is performed before differentiating.

This simple observation can be used to extend the averaging principle also to planet crossing orbits: we have shown [Gronchi and Milani 1998] that in the planet crossing case, even if equations (2) have no sense, it is possible to define piecewise differentiable solutions of equations (3). The right hand sides are always defined inside a circular domain in the space $(e\cos\omega, e\sin\omega)$, except for the arcs of the node crossing lines that lie in this domain, where the derivatives of R have a twofold definition (see Figure 1), so that there is discontinuity of the first kind in the velocity of the solutions at the node crossing points.

The main idea of the theory is to use a singularity extraction method, proposed long ago by Kantorovich [Demidovic 1966], to compute the derivatives;

for instance, if the two orbits intersect at the ascending mutual node and if the differentiation variable is the eccentricity, so that we have to compute

$$\frac{\partial}{\partial e}\int_T \frac{1}{D}\, d\ell\, d\ell'$$

(where $T = [-\pi,\pi]\times[-\pi,\pi]$ is the torus of the two anomalies, and $D = |\vec{x}-\vec{x}'|$ is the distance from the planet) we employ the decomposition

$$\frac{\partial}{\partial e}\int_T \frac{1}{D}\, d\ell\, d\ell' = \frac{\partial}{\partial e}\int_T \frac{1}{d}\, d\ell\, d\ell' + \frac{\partial}{\partial e}\int_T \left[\frac{1}{D}-\frac{1}{d}\right] d\ell\, d\ell' \tag{4}$$

where the function d is obtained by an approximation due to Wetherill [Wetherill 1967]: it represents the distance between two straight lines that approximate the motion on the two orbits of the problem in a neighborhood of the ascending mutual node (see Figure 2). d^2 is a quadratic form in the variables (ℓ,ℓ'). This approximation is used in a neighborhood of an ascending node orbit crossing. The analogous quantity d_1 can be defined and used in a neighborhood of a descending node crossing.

The choice of this approximate distance allows to extract the principal part of the singularity from the derivative $\frac{\partial}{\partial e}\frac{1}{D}$ and the *derivative of the remainder function*

$$\frac{\partial}{\partial e}\left[\frac{1}{D}-\frac{1}{d}\right]$$

has a first order polar singularity; so it is integrable and, by the already cited *theorem of differentiation under the integral sign*, its integral average

$$\int_T \frac{\partial}{\partial e}\left[\frac{1}{D}-\frac{1}{d}\right] d\ell\, d\ell' \tag{5}$$

corresponds to the *derivative of the averaged remainder function*

$$\frac{\partial}{\partial e}\int_T \left[\frac{1}{D}-\frac{1}{d}\right] d\ell\, d\ell'$$

which appears in equation (4). Figure 3 illustrates the removal of the singularity in one of the partial derivatives of the perturbing function. In this example the removal is achieved for the singularity introduced by the collision with the Earth, another similar procedure would be required to remove the other singularity due to Venus.

The first term in the right-hand side of equation (4) has the same discontinuity properties as the derivative of $\overline{R}$, but the advantage of this decomposition is that the computation of

$$\frac{\partial}{\partial e}\int_T \frac{1}{d}\, d\ell\, d\ell'$$

can be performed analytically. This computation is described in details in [Gronchi and Milani 1998]; it can be summarized in the following two steps.

First, the analytical integration of $1/d$ is performed by switching to polar coordinates (r, θ); the result is the sum of elliptic integrals plus an explicit term under square root:

$$\int_T \frac{1}{d}\, d\ell\, d\ell' = 2\Delta \cdot \left[\sum_{i=1}^{3} \int_{\theta_{i-1}}^{\theta_i} \sqrt{\tilde{C} + r_i^2(\theta)}\, d\theta - \pi\sqrt{\tilde{C}} \right] \tag{6}$$

where $\sqrt{\tilde{C}}$ represents the minimum distance between the two straight lines of Wetherill approximation (see [Gronchi and Milani 1998] for the definition of all the symbols). In this formula the only term which is not differentiable with respect to the orbital elements is the last one containing $\sqrt{\tilde{C}}$:

$$\sqrt{\tilde{C}} = \left|d^+_{nod}\right| \cdot \left[1 - \frac{a^2 a'^2 e^2 \sin^2\omega}{(1-e^2)det(\underline{\underline{\mathbf{A}}})}\right]^{\frac{1}{2}} \tag{7}$$

because it contains the absolute value of the distance at the ascending node $|d^+_{nod}|$, not differentiable at the node crossing points. The Keplerian orbital elements refer to the orbit of the asteroid, of the planet when primed; $\underline{\underline{\mathbf{A}}}$ is the matrix associated to the quadratic form d^2. Note that this formula is always applicable, even for zero inclination, with the only exception of orbits both coplanar and tangent at perihelion/aphelion. Formula (7) applies near the ascending node; at the other node, d^+_{nod} should be replaced by d^-_{nod}, the distance at the descending node .

Second, the derivatives (with respect to the orbital elements e, ω, I) are applied to this analytical formula; this is an essential step for the theory, because it allows to write analytical formulas for the discontinuity of the derivatives of the perturbing function (see Appendix A). This has an essential role in our numerical integration scheme (see Section 3), allowing to "jump over" the singularity at the node crossing line.

The derivatives, to be used in the right hand side of Hamilton equations (3), are computed by quadrature of the elliptic integrals and by explicit computation of the analytical derivatives of $\sqrt{\tilde{C}}$. This formula is applicable only outside the node crossing lines, but, unlike the conventional double integral, is numerically stable on both sides of the node crossing lines.

2.3 The double crossing case

We have extended the theory to have the possibility of efficiently computing the secular evolution around a *double node crossing*, that is a simultaneous crossing both at ascending and descending node with the same planet. In fact we have detected cases with almost exact double crossings in more than one case of real asteroids.

There is even a real asteroid whose initial conditions are almost double crossing at the present time: the recently discovered 1999 AN_{10}, whose short period dynamics has been intensively studied [Milani et al. 1999]. This has important implications, in particular in the persistence of impact risk for an extended time span (several centuries).

Here we present briefly the changes to our theory that allow the extension to the *double crossing* case.

Because of the double periodic nature of the true distance function D, the integration domain covering the torus T can be shifted arbitrarily. However, it is essential to make sure that the singularities of collision do not occur on the boundary of the integration domain, otherwise a "phantom" singularity would appear.

A double crossing can occur only for $\omega = \pi/2, 3\pi/2$, that is for $\cos\omega = 0$. The eccentric anomaly u corresponding to the passage at the nodes satisfies the equations

$$\cos u = \frac{e \pm \cos\omega}{1 \pm e\cos\omega}$$

and for $\omega = \pi/2, 3\pi/2$ this implies $\cos u = e < 1$, which is incompatible with $u = 0$, which corresponds to $\ell = 0$. Thus the interval $[0, 2\pi]$ for ℓ is such that the singularity of collision would not occur on the boundary. For ℓ', if the origin of the planetary orbital phase is taken to be the ascending node of the asteroid orbits, collisions can only occur at $\ell' = 0, \pi$. Thus the interval $[-\pi/2, 3\pi/2]$ for ℓ' does not have singularities on the boundary.

In a neighborhood of a double crossing the approximated inverse distance near the ascending node $1/d$ is used along with the other one $1/d_1$, applicable near the descending node. The difference $1/D - 1/d - 1/d_1$ is shown to be limited and with integrable derivatives. Note that this is true only if "phantom" singularities are avoided, because $1/d$ and $1/d_1$ are not periodic functions. Then the computations can proceed as in the single crossing case.

3 The averaged numerical integration

The numerical method we use to solve Hamilton equations (3) is an implicit Runge-Kutta-Gauss algorithm of order 6, which is also symplectic [Sanz-Serna 1988].

In a previous work we used a simpler algorithm to compute the time evolution of an Earth crossing asteroid without removal of the singularities of collision [Milani and Gronchi 1999], but the procedure was not reliable, depending in an unstable way upon the nodal distance at each Runge-Kutta substep. Thus it was possible to compute proper elements only for one asteroid at a time, and with comparatively frequent human interventions in the computational procedure. To compute proper elements for hundreds of NEA's we need numerically stable algorithms and fully automated, reliable software.

The method of Kantorovich, besides being an essential tool in the theoretical proof of the existence of the generalized solutions, is employed in this work in the numerical integration algorithm when an orbit is very close to a node crossing. This stabilizes the computation even when the nodal distance is small and the averaging integrals are "almost improper", that is with very sharp extrema.

However, also in the present algorithm the right-hand side of equations (3) are never computed at the node crossing points, where they are defined in a twofold way. This is possible because the Runge-Kutta-Gauss methods use substeps which include neither the starting point nor the final point of the step being computed. We resort to the following procedure (which had already been used in [Milani and Gronchi 1999]): every time the asteroid orbit is close enough to a node crossing line, the standard iteration scheme known as *regula falsi* is used to set the second extreme point of the step exactly on that line; as the computation of the right-hand sides is performed only in the intermediate points, the computation at node crossings is avoided.

When the node crossing point is reached within the required precision, the integration needs to restart from there for the next step. In the implicit Runge-Kutta schemes the following step restarts the iterations (to solve the implicit equations) from a first guess, obtained by polynomial extrapolation, of the values used in the previous step. This would fail in this case, because of the discontinuous jump of the right-hand side of (4); we apply to the first guess, extrapolated from values computed before the jump, a correction computed by means of the explicit analytical formula for the jump of the derivatives of $\overline{R}$ (see Appendix).

There are a number of technicalities involved in the regula falsi method to arrive at the node crossing line exactly at the end of one integration step. This is necessary to ensure that the computational procedure is stable under all possible conditions: to compute proper elements for a large catalog of orbits requires much more reliability than to experiment with just a few cases, as in our previous work [Milani and Gronchi 1999]. The main problem is that, the right hand side being discontinuous on the node crossing line, the integration steps need to be controlled to be rather shorter than longer than the exact time span to node crossing, to avoid running into the discontinuous behavior. The regula falsi iterations can start only when a change in the sign of the nodal distance takes place between the last intermediate point of the step and the beginning of the next step (with the symbols used in Figure 4, the node crossing line must be between the last cross and the next circle). If a change of sign already occurs in the intermediate points, the stepsize is halved. Even when no change of sign occurs in the current step, an extrapolation algorithm is used to predict whether a node crossing could take place in the next step. For this purpose, the stepsize to crossing is computed in two ways, by regula falsi, that is by a linear extrapolation, and by polynomial extrapolation; the minimum of the two is taken as a prudent estimate of the stepsize to crossing. If the stepsize to crossing is less than the stepsize used in the non crossing portion of the integration, the stepsize of the integration is adjusted to this prediction, to avoid "overshooting" of the node crossing line.

A separate regula falsi algorithm is used to compute the value of the orbital elements solution of the averaged equations exactly at the symmetry lines (which are $\omega = 0, \pi/2, \pi, 3/2\pi$); the computation of the proper elements requires, at least for the circulation and symmetric libration cases (see Section 5), to compute the solution of the averaged equations between two successive crossings of the symmetry lines. The complete secular evolution is obtained by means of the symmetries $\omega \longmapsto \omega + \pi$ and $\omega \longmapsto -\omega$.

The computation of the proper frequencies requires some additional comment. Let us suppose we have, from the output of the numerical integration of the averaged equations, the times t_0 and $t_{\pi/2}$ at which $\omega = 0$ and $\omega = \pi/2$, respectively, and the corresponding values Ω_0 and $\Omega_{\pi/2}$ of the longitude of the node. The secular frequency of the argument of perihelion is computed taking into account that in this case ω must be circulating: because of the symmetries, the period of this circulation must be four times the time interval required by an increase by $\pi/2$:

$$g - s = \frac{2\pi}{4(t_{\pi/2} - t_0)} \ .$$

The proper frequency s of the node has to be computed taking into account that the node has a secular precession, but also long periodic oscillations controlled by the argument 2ω. The averaged Hamiltonian does not contain Ω, but only the cosine of 2ω, thus the perturbative equation of motion for Ω is

$$\frac{d\Omega}{dt} = -\frac{\partial \overline{R}(2\omega)}{\partial Z} \ ;$$

if ω is circulating, then it is possible to change (at least locally) independent variable and write the equation directly linking Ω to ω:

$$\frac{d\Omega}{d\omega} = \frac{\partial \overline{R}/\partial Z}{\partial \overline{R}/\partial G} = F(\cos(2\omega))$$

with a right hand side containing only $\cos(2\omega)$. By quadrature, the solution for Ω as a function of ω is:

$$\Omega(\omega) = \alpha\ \omega + f(\sin(2\omega))$$

and the function $f(\sin(2\omega))$ is zero for $\omega = 0, \pi/2$, thus

$$\Omega_{\pi/2} - \Omega_0 = \alpha\ \pi/2$$

identifies the secular part of the evolution of Ω, whose frequency can be thus computed simply by

$$s = \frac{\Omega_{\pi/2} - \Omega_0}{t_{\pi/2} - t_0} \ .$$

The symmetric libration cases are different in that ω returns to the same multiple of $\pi/2$ after a time interval Δt, while in the same time span Ω has changed by $\Delta\Omega$. Then $g - s$ is not a proper frequency (its secular part is by definition zero), and by arguments similar to the ones above, the libration frequency lf and the secular frequency of the node s can be computed by

$$lf = \pm\frac{2\pi}{2\Delta t} \qquad s = \frac{\Delta\Omega}{\Delta t}$$

where the sign has to be chosen negative for clockwise libration.

In the asymmetric libration case the procedure to terminate the numerical integration and to compute the proper frequencies would have to be different, but, as discussed in Section 5, this case did not occur in the proper elements catalogs we have computed so far.

4 The proper elements catalog

This generalized averaging principle has been used to compute, by a model including five planets (from Venus to Saturn) in circular, coplanar orbits, the secular evolution of 987 asteroids from the database of NEA orbits of NEODyS, which is available at the WWW address:

`http://newton.dm.unipi.it/neodys/`

From the output of this secular propagation we have computed proper elements and proper frequencies: the results of this computation are available through NEODyS, and also at the following anonymous ftp address:

`ftp://copernico.dm.unipi.it/pub/propneo/`

There is a catalog file containing the following data:

1) the asteroid name,

2) the initial condition date (in Julian days),

3) the prime integrals of the theory, that is the semimajor axis a, the averaged perturbing function $\overline{R}$ and Z, which is the third component of the angular momentum. The qualitative behavior of ω (circulation or libration) is also specified.

4) the number of crossings with each planet (Venus to Saturn, numbered from 2 to 6) per period of ω.

5) the proper frequencies $g - s$ of ω, s of the longitude of the node Ω and g of the longitude of perihelion ϖ. In case ω is librating $g - s = 0$ by definition, and we give the frequency lf of the libration argument.

6) the proper elements $e_{min}, e_{max}, I_{min}, I_{max}$ (respectively minimum and maximum of eccentricity and inclination over one period of ω) and, in the libration case, also $\omega_{min}, \omega_{max}$ (the minimum and maximum of the value of ω) and the amplitude of the libration.

7) for each Earth crossing we have also computed the *encounter conditions*, namely the four quantities $(U, \theta, \phi, \lambda)$: they are a combination of the orbital elements computed at node crossings [Valsecchi et al. 1999]. These conditions have the property of being directly observable during a close approach. U represents the normalized object velocity in a geocentric reference frame; θ and ϕ are two angles defined by the geocentric velocity vector, related to the *radiant* in the case of a meteor stream; λ is the longitude of the crossing point along the Earth orbit, that is the date of the crossing.

The proper elements, especially the semimajor axis a, the range of values of the eccentricity and the inclination are the best way to describe the medium term (1 000 to 100 000 years) evolution of the orbit. The proper frequencies are used to detect secular resonances: when the frequencies g and s are close to some of the fundamental frequencies of the planetary orbits (g_i and s_i respectively), then the long term behavior can be dominated by secular resonance effects and the theory, as presented here, cannot provide stable proper elements. The encounter conditions are the same quantities which are directly observable for meteors; their computation for asteroids whose orbit is not crossing the Earth at the present time, but can cross at some time in the future and in the past, can be used in the search for parent bodies of meteor streams. The same variables contain the information on the average probability of impact over a long time span. These data are to be used also as input to study the fine structure of the sequences of close approaches taking place near node crossings [Valsecchi et al. 2000].

5 Qualitative dynamical behavior

The well known asteroid classification in Apollo (perihelion inside Earth's orbit, $a > 1$), Amor (perihelion outside Earth's orbit but at no more than 1.3 AU) and Athen ($a < 1$, aphelion outside Earth's orbit), is useful as a classification of instantaneous, osculating orbital elements, but is by no means stable for time spans of $100, 1\,000$ years or more [Milani et al. 1989]. Note that in the context of the approximation used in this paper, namely the planets on circular orbits, the boundary between Apollo and Amors needs to be set at a perihelion distance of exactly 1 AU for consistency.

Over a longer time span, conditions of the form

$$a(1-e) < a' < a(1+e)$$

cannot assure the existence of crossings, unless we use the simplistic assumption that the argument of perihelion ω precesses without changes in eccentricity and inclination; therefore the *Apollo + Athen* class, defined on the basis of today's osculating elements, is not equivalent to the class of Earth crossing asteroids, that are by definition the asteroids whose orbit crosses the Earth's during their secular evolution. Of course an Amor has to become later an Apollo to undergo a node crossing with the Earth.

As a result of the full secular evolution, we find more complicated behaviors than the simple quadruple crossing Apollo/Athen and the simple non-crossing Amor. This results, e.g., from the $e - \omega$ coupling (the minimum of e, occurring when $\omega = 0, \pi$, is such that $a(1-e) > a'$). There is also the case in which the argument of perihelion ω librates around either $\omega = \pi/2$ or $\omega = 3\pi/2$; in this case an asteroid can be an Apollo during a fraction of the ω cycle, and still never cross the orbit of the Earth (*Kozai class* dynamics, [Milani et al. 1989]). In other cases, the number of crossings can be 8 per period of ω; for the Athen asteroids, libration around $\omega = 0, \pi$ can occur [Michel and Froeschlé 1997].

In Figure 5 we have plotted in the phase space (current e, a) the orbital elements of the 987 objects examined and we have enhanced the ones that, during their time evolutions, do not behave like their initial classification in Apollo, Amor and Athen should suggest: indeed, we have found 94 Apollo (out of 504) which are not Earth crossing (crosses), 11 Amor (out of 408) which are Earth crossing (circles) and only 3 Athen (out of 75) not Earth crossing (circled crosses).

We can observe that in Figure 6 the *non Earth crossing Apollo* have moved outside the Earth crossing area while in Figure 7 the *Earth crossing Amor* have moved inside it.

In our integration we have detected two of the main qualitative behaviors in the (ω, e) plane [Gronchi and Milani 1999]: *direct circulation* and *counter-clockwise symmetric libration of perihelion argument.* No asymmetric libration (that is a libration whose center is not an integer multiple of $\omega = \pi/2$) has been found in this sample of orbits, although we know it could occur for comet/Centaur type orbits.

In the following figures there is a sample of the secular evolution of the NEA's in the (ω, e) plane; the level lines of the averaged perturbing function are plotted and the time evolution of the objects considered is enhanced by crosses along the curve representing the constant value of their perturbing function. Only a portion of the evolution is plotted here, but this is enough to compute by symmetry [Gronchi and Milani 1999] the evolution at any time (within the validity of this theory).

In Figure 8 there are four examples of circulation of ω: the first object, 1991 TT is an Apollo asteroid that is preserved from Earth crossings by a $e - \omega$ coupling mechanism, that is the eccentricity oscillates and reaches its minimum value at the integer multiples of π (when the apsidal line corresponds to the mutual node line); the second asteroid, 1991 BB, is an ordinary quadruple crosser (crosses the Earth four times for period of revolution of ω) while the third, 1994 JX, is one of the 11 Amor asteroids that become crosser during their evolution (it is even an octuple crosser). The last case, 1998 $FG12$, is peculiar because it crosses the Earth orbit in a *tangent way*, staying in a neighborhood of the crossing curve for a very long time; this object will have a comparatively high probability of impacting the Earth, but this will happen during the *fifth millennium AD.*

In Figure 9 there are four examples of librating objects. The first is one of the 3 Athen asteroid preserved from crossing the Earth orbit: the symmetric libration around $\omega = 2\pi$ ensures its safety. On the contrary, the libration in the second example, (2102) Tantalus, forces it to cross the orbits of Venus and the Earth, both four times for each period of the libration argument.

The dynamics of the asteroid 1999 AN_{10} has been extensively studied because of its very small distance to the orbit of the Earth at both nodes. When fewer observations were available, it was not possible to exclude an impact with our planet in the next century. As figure 9 shows, the secular evolution is now very close to a double crossing at both the ascending and descending node with the Earth; the period of libration of ω is very long, this implies that the hazardous proximity of this orbit to the orbit of our planet will be maintained for centuries. In fact precisely the interest for this case forced us to revise our algorithm to make it efficient and stable also in the double crossing case, as discussed in Section 2.3.

The last asteroid of figure 9, 1997 NC_1, has a very strange averaged dynamics: the level curve of the averaged Hamiltonian, to which its motion is bounded, is very close to the separatrix between libration and circulation, so that, while its perihelion argument in the long run circulates, its orbit draws an arc of libration near the integer multiples of π.

6 Secular resonances

From the computation of the proper frequencies it has been possible to draw a map of the dynamical structure of the Near Earth Asteroids, in particular identifying the location of the most important secular resonances.

In the plane (minimum e, a) we have plotted all the 987 NEA; we have marked with a circle the ones which we expect would be affected by the resonance ν_6 (that is $g - g_6 \simeq 0$), with a square the ones by the resonance ν_{16}, (that is $s - s_6 \simeq 0$), and with a cross the ones by the resonance ν_5 (that is $g - g_5 \simeq 0$). We have used a simple criterion, marking the ones with the secular divisor associated to the resonance smaller than 2 arcsec per year. A better map should be made by using actual estimates of the resonance widths, but these values are reasonable, given the results of the numerical integrations such as [Gronchi and Michel 2000].

The well known transport mechanism towards high eccentricities due to the ν_6 resonance is clearly visible in Figure 10. It had already been studied by a secular theory and by numerical integrations in the main belt [Morbidelli and Henrard 1991, Froeschle and Scholl 1986]. Semianalytic theories confirmed its existence also near the Earth crossing region [Michel 1997, Michel and Froeschlé 1997], which could not be studied by the same theory for the singularities appearing in the perturbing function.

Also the region of the Hungaria can be noticed: it is characterized by the presence of the ν_{16} resonance.

In addition to these already known dynamical structures determined by secular resonance, a new one appears from this plot: looking at the crosses also the ν_5 resonance seems to play a role in the increase of the eccentricity, mostly inside the Earth crossing region. This is consistent with the identification of the same resonance for orbits near-Earth but not crossing, as obtained by a semianalytic theory [Michel and Froeschlé 1997]. We are not aware of a numerical confirmation of this resonant effect for the orbits of NEA; this confirmation would be useful, and even more useful would be an assessment of the possible role of the ν_5 resonance in injecting already Earth-crossing objects in orbits impacting the Sun. However, [Gladman 1997] detected this resonance in a simulation of the transport of meteorites from Mars to the Earth.

7 Conclusions and future work

We have defined and implemented a new, numerically stable and efficient algorithm to compute the secular evolution of planet crossing orbits. The output of this computation for all known Near Earth Asteroids is now available online. This is an important step forward, since the collision singularity previously prevented this computation. However, we are well aware that this has to be considered just a first step. The following are the main limitations of this theory, and our future work will try to remove them.

First, the averaging with respect to the fast angle variables ℓ, ℓ' is applicable provided there are no mean motion resonances. A strong mean motion resonance (e.g., 3:1 with Jupiter) can result in *super-crossing* behavior, typical of the Alinda dynamical class, [Milani et al. 1989], which implies that the averaged solution might even fail to detect node crossings which can indeed occur. We should detect the occurrence of strong resonances and either remove the corresponding proper elements, which are not reliable, from the catalog, or use a different definition and a different algorithm, possibly along the lines of [Milani and Baccili 1998].

Second, this theory is of order zero in the eccentricities and inclinations of the perturbing planet; an extension to at least order one in these small quantities would be necessary. This would significantly improve the accuracy, hence the stability in time of the proper elements, and also improve the secular frequencies and allow to better detect the orbits affected by secular resonances.

Third, the very occurrence of crossings, the encounter conditions, and especially the times of the node crossings are significantly affected by the eccentricity of the target planet; even the Earth's current eccentricity of 0.017 is not negligible. We have not yet used our proper elements for computations of probabilities of impact (averaged over one period of ω) because we suspect that some correction accounting for the eccentricity of the Earth would be necessary.

Fourth, the proper elements should be computed starting from mean elements, that is after removing the short periodic perturbations.

The accuracy of this theory, e.g., in the estimation of the frequencies g, s, lf, and the stability of these proper elements with time, need to be tested by comparison of the averaged solutions with some average of the full solutions with different initial anomalies. A first set of such tests is discussed in [Gronchi and Michel 2000]; as this theory progresses, by removing some of the above limitations, similar tests should be repeated to measure the progress in the accuracy and reliability.

8 Appendix

This appendix contains the analytical formulas representing the discontinuity of the derivatives of the averaged perturbing function $\overline{R}$.

If we choose, for instance, the direction from the $\{d^+_{nod} > 0\}$ to the $\{d^+_{nod} < 0\}$ area, the difference in the twofold definition of the derivatives of the averaged perturbing function are

$$\mathrm{Diff}\left(\frac{\partial\overline{R}}{\partial G}\right) = \frac{k\mu(1-e^2)}{2\pi\sqrt{det(\underline{\underline{\mathbf{A}}})}\cdot e\sqrt{a}} \cdot \left[1 - \frac{a^2a'^2e^2\sin^2\omega}{(1-e^2)det(\underline{\underline{\mathbf{A}}})}\right]^{\frac{1}{2}} \cdot \mathrm{Diff}\left(\frac{\partial|d^+_{nod}|}{\partial e}\right)$$

$$\mathrm{Diff}\left(\frac{\partial\overline{R}}{\partial \omega}\right) = -\frac{k^2\mu}{2\pi\sqrt{det(\underline{\underline{\mathbf{A}}})}} \cdot \left[1 - \frac{a^2a'^2e^2\sin^2\omega}{(1-e^2)det(\underline{\underline{\mathbf{A}}})}\right]^{\frac{1}{2}} \cdot \mathrm{Diff}\left(\frac{\partial|d^+_{nod}|}{\partial \omega}\right)$$

$$\mathrm{Diff}\left(\frac{\partial\overline{R}}{\partial Z}\right) = 0$$

where

$$det(\underline{\underline{\mathbf{A}}}) = \frac{a^2a'^2}{(1-e^2)} \cdot (\sin^2 I(1+2e\cos\omega+e^2) + e^2\cos^2 I\sin^2\omega)$$

$$\mathrm{Diff}\left(\frac{\partial}{\partial e}|d^+_{nod}|\right) = \left[\frac{\partial^+|d^+_{nod}|}{\partial e} - \frac{\partial^-|d^+_{nod}|}{\partial e}\right] = \frac{2a(\cos\omega(1+e^2)+2e)}{(1+e\cos\omega)^2}$$

$$\mathrm{Diff}\left(\frac{\partial}{\partial \omega}|d^+_{nod}|\right) = \left[\frac{\partial^+|d^+_{nod}|}{\partial \omega} - \frac{\partial^-|d^+_{nod}|}{\partial \omega}\right] = -\frac{2a(1-e^2)e\sin\omega}{(1+e\cos\omega)^2}$$

(see [Gronchi and Milani 1998] for the notation).

In case of crossing at the descending node, we have to modify the previous formulas as follows:

$$det(\underline{\underline{\mathbf{A}}}) = \frac{a^2a'^2}{(1-e^2)} \cdot (\sin^2 I(1-2e\cos\omega+e^2) + e^2\cos^2 I\sin^2\omega)$$

$$\mathrm{Diff}\left(\frac{\partial}{\partial e}|d^-_{nod}|\right) = \frac{\partial^+|d^-_{nod}|}{\partial e} - \frac{\partial^-|d^-_{nod}|}{\partial e} = -\frac{2a(\cos\omega(1+e^2)-2e)}{(1-e\cos\omega)^2}$$

$$\mathrm{Diff}\left(\frac{\partial}{\partial \omega}|d^-_{nod}|\right) = \frac{\partial^+|d^-_{nod}|}{\partial \omega} - \frac{\partial^-|d^-_{nod}|}{\partial \omega} = \frac{2a(1-e^2)e\sin\omega}{(1-e\cos\omega)^2}$$

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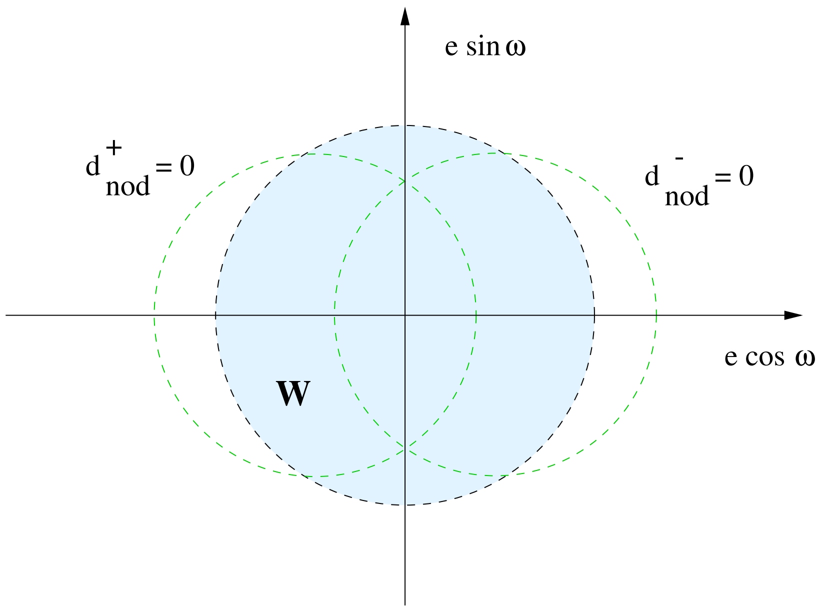

Figure 1: The definition domain W of the averaged Hamilton equations in the phase space $(e\cos\omega, e\sin\omega)$: the node crossing line with one planet is also plotted, both at the ascending and the descending node (labeled by $d^+_{nod} = 0$ and by $d^-_{nod} = 0$ respectively). The outer boundary of the domain is the circle $e = e_{max}$, with e_{max} a function of the integral Z.

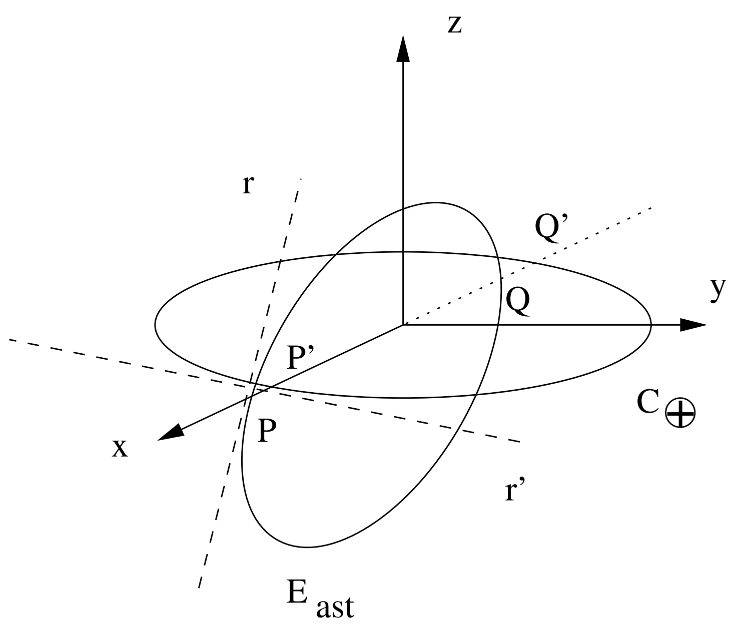

Figure 2: Wetherill approximation for the ascending mutual node: $C_{\oplus}$ represents the Earth circular orbit and E_{ast} represents the asteroid elliptic Keplerian orbit; the pairs of points (P, P') and (Q, Q') are the ascending and descending mutual nodes respectively. The straight lines r and r' are tangent to the ellipse and to the circle in P and P' respectively.

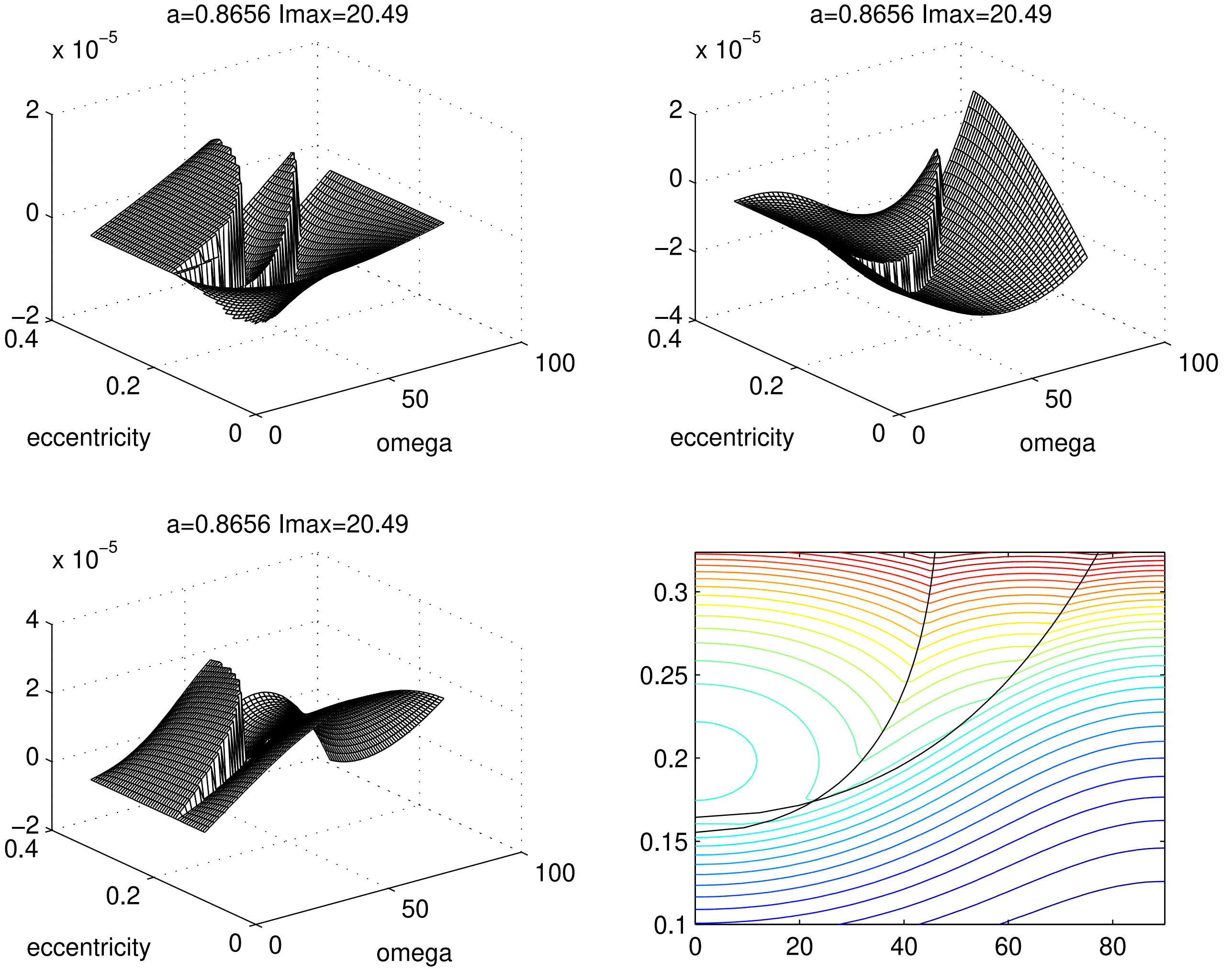

Figure 3: An illustration of the discontinuity extraction procedure. Top left: the graphic of the derivative of the integral average of $1/D$ with respect to ω, as a function of ω, e; there are two lines of discontinuity, corresponding to crossings at the ascending node with the Earth and at the descending one with Venus. Top right: the derivative of the integral average of $1/d$ with respect to ω; note the discontinuity at the ascending node with the Earth. Bottom left: the difference between the two previous functions shows that the discontinuity due to the Earth has been removed; note that the difference function is continuous, but not differentiable along the ascending node crossing line. The discontinuity at the descending node with Venus is still present. Bottom right: the level curves of the integral average of $1/D$.

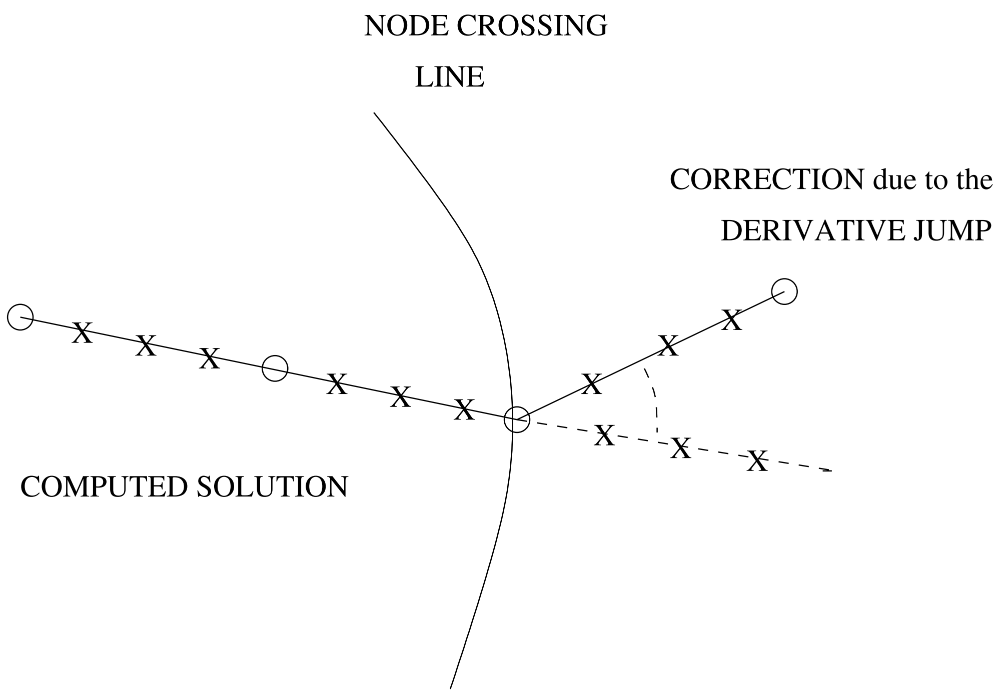

Figure 4: Graphical description of the algorithm employed in this numerical integration: it is an implicit Runge-Kutta-Gauss method, symplectic, of order 6. In an integration step, delimited by two consecutive circles, the only points in which the derivatives of $\overline{R}$ are computed are the 3 intermediate points marked with crosses. When the node crossing line is reached, at the end of one integration step, a correction taking into account the explicitly computed jump of the derivatives is added to the right-hand sides of the equations.

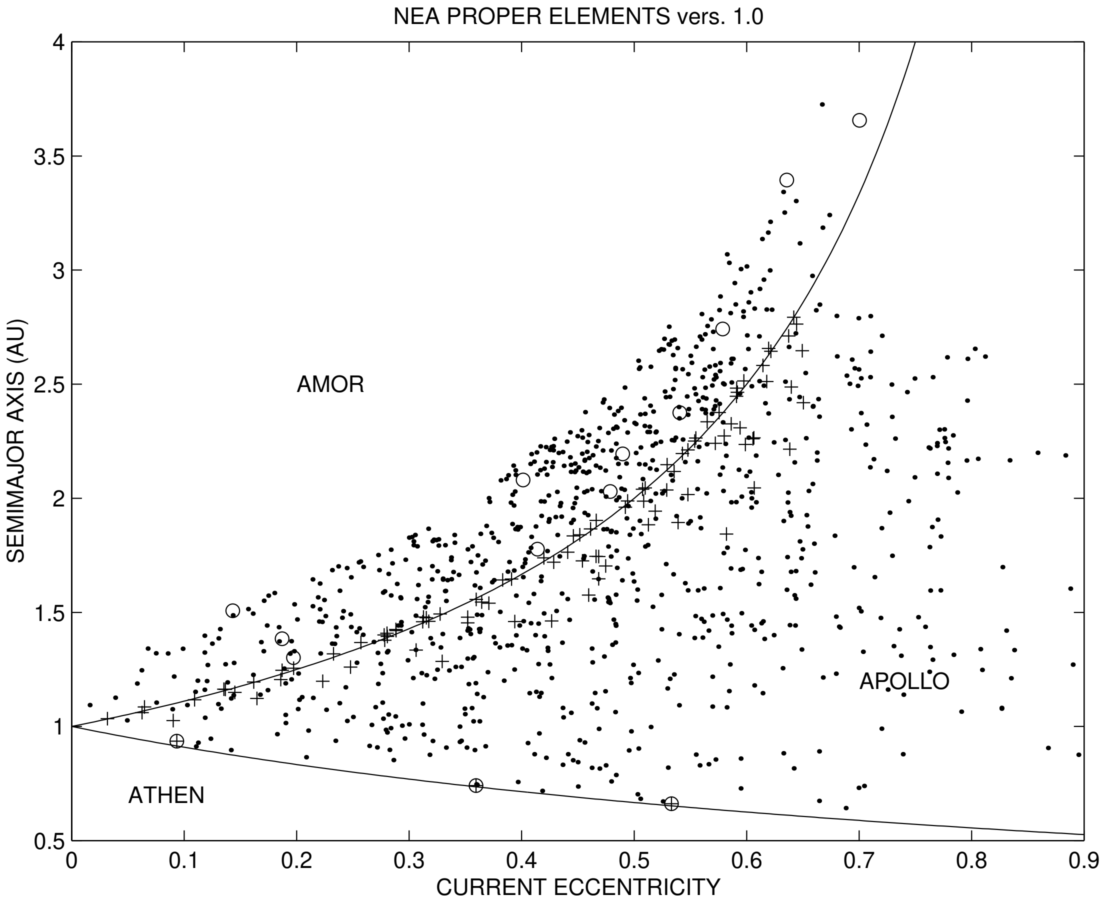

Figure 5: In the phase space (current e, a) we have plotted 987 asteroids drawing with crosses the 94 not Earth crossing Apollo, with circles the 11 Earth crossing Amor and with circled crosses the 3 not Earth crossing Athen. The two curves plotted represent $a' = a(1-e)$ and $a' = a(1+e)$ so that the *Earth crossing area* is between them.

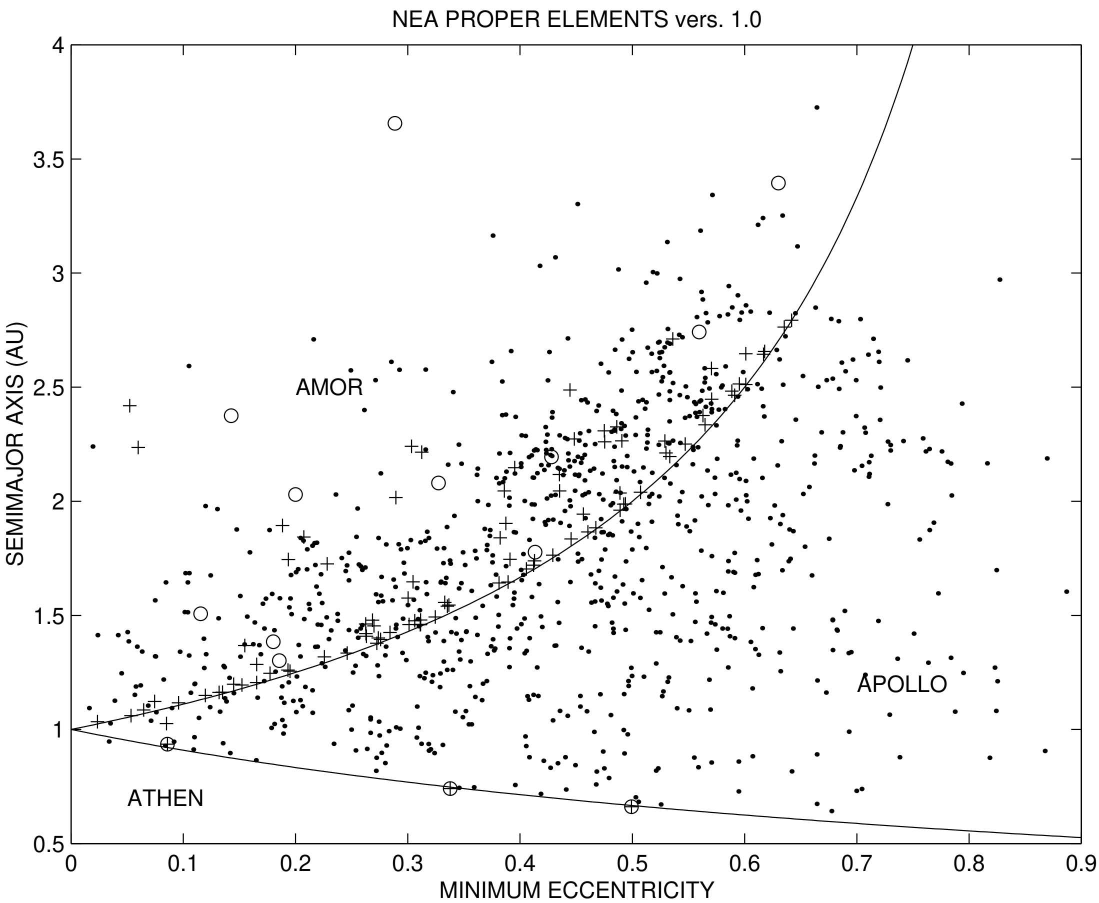

Figure 6: In the phase space (minimum e, a) some Apollo asteroids have moved in the non crossing area; these *non Earth crossing Apollo* are marked with a cross.

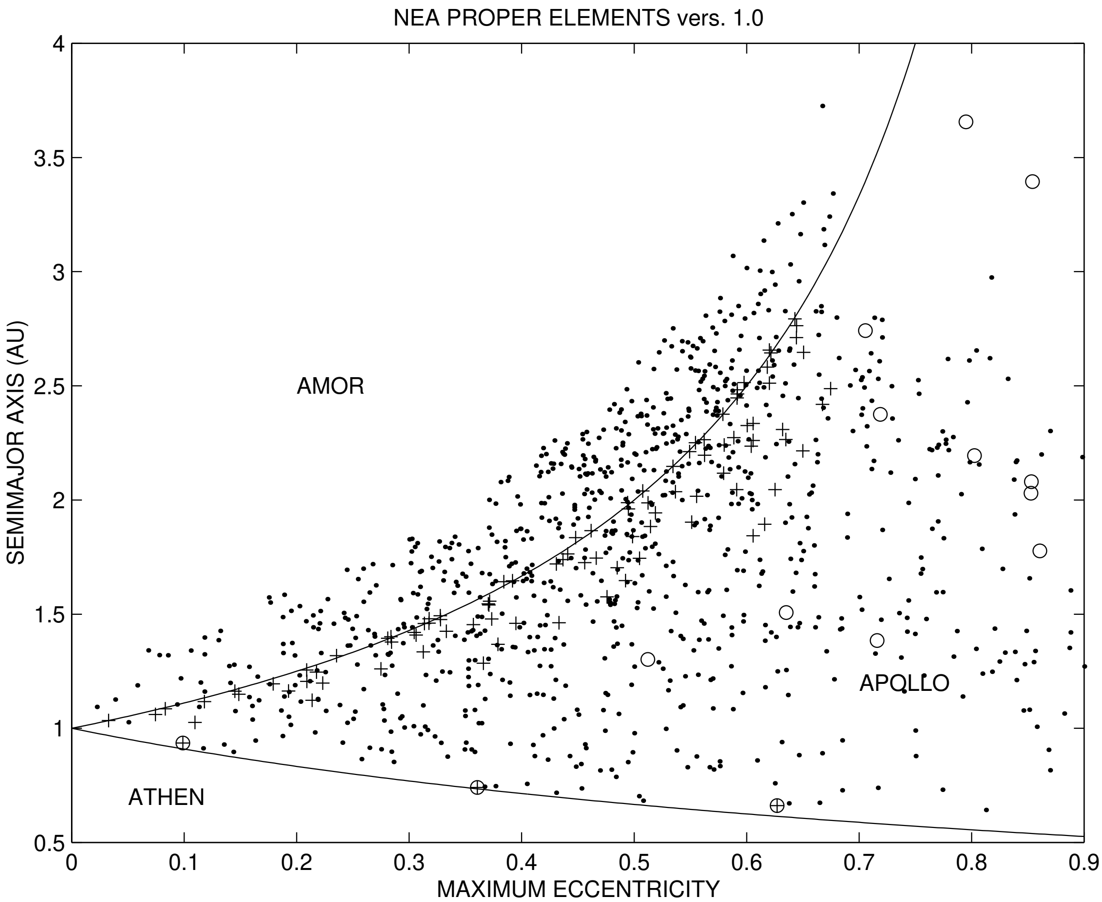

Figure 7: In the phase space (maximum e, a) 11 Amor asteroids have moved in the Earth crossing region; these *Earth crossing Amor* are marked with a circle.

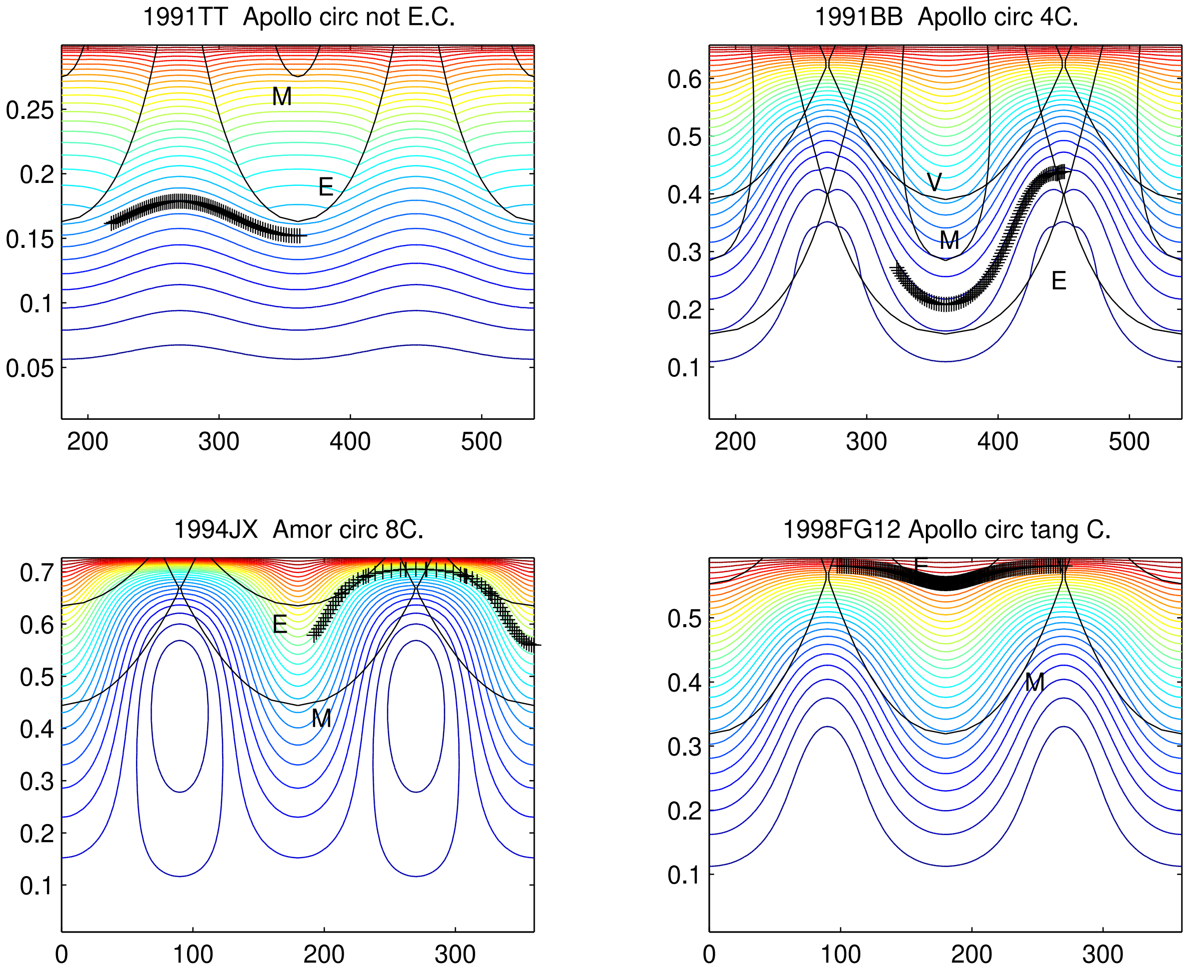

Figure 8: Time evolution of Near Earth Asteroids whose perihelion argument is circulating; the complete evolution of these objects can be obtained from the partial one, plotted here, employing the symmetries of the averaged perturbing function in the plane (ω, e). The node crossing lines with the Earth (E), Mars (M) and in some cases with Venus (V) are also shown.

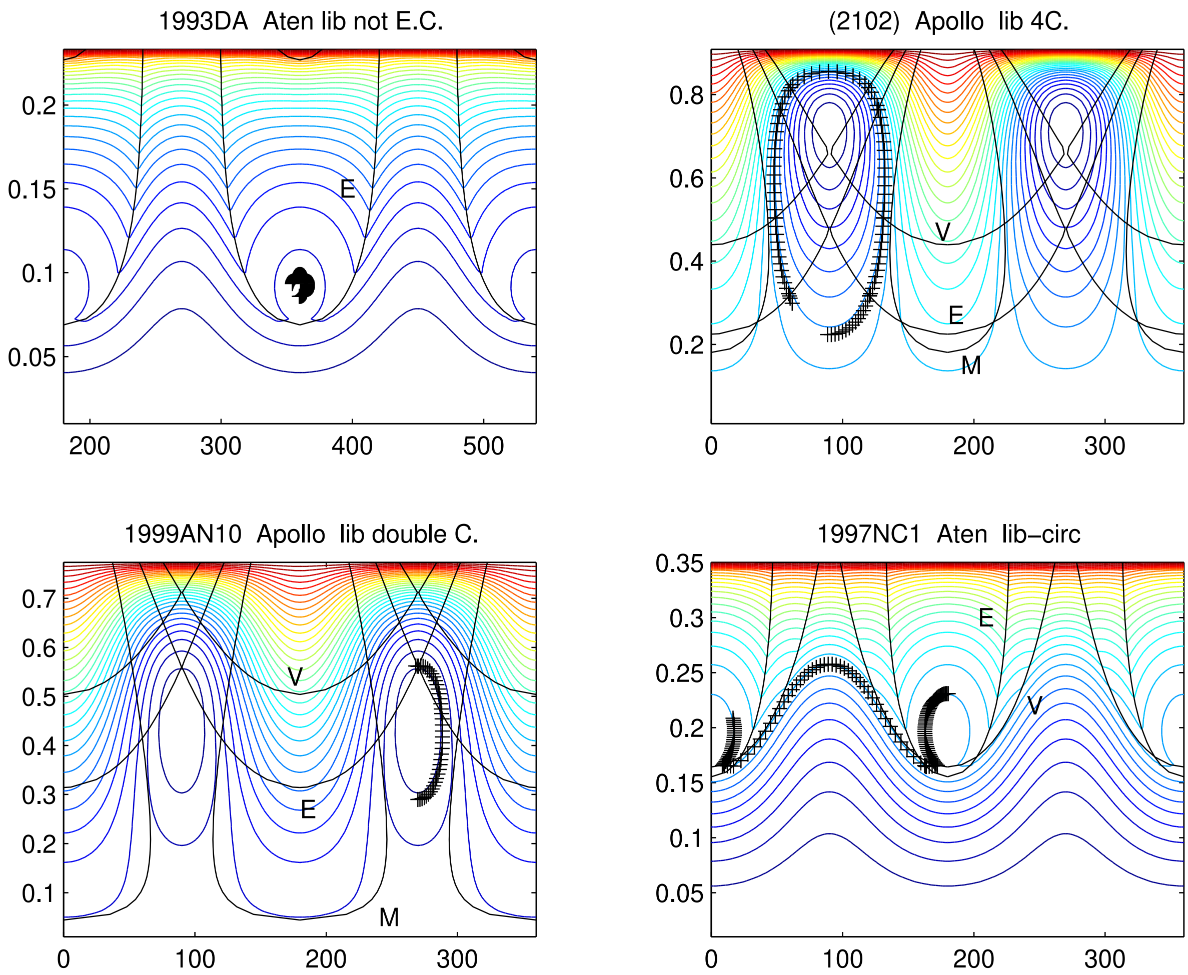

Figure 9: Time evolution of Near Earth Asteroids whose perihelion argument is librating.

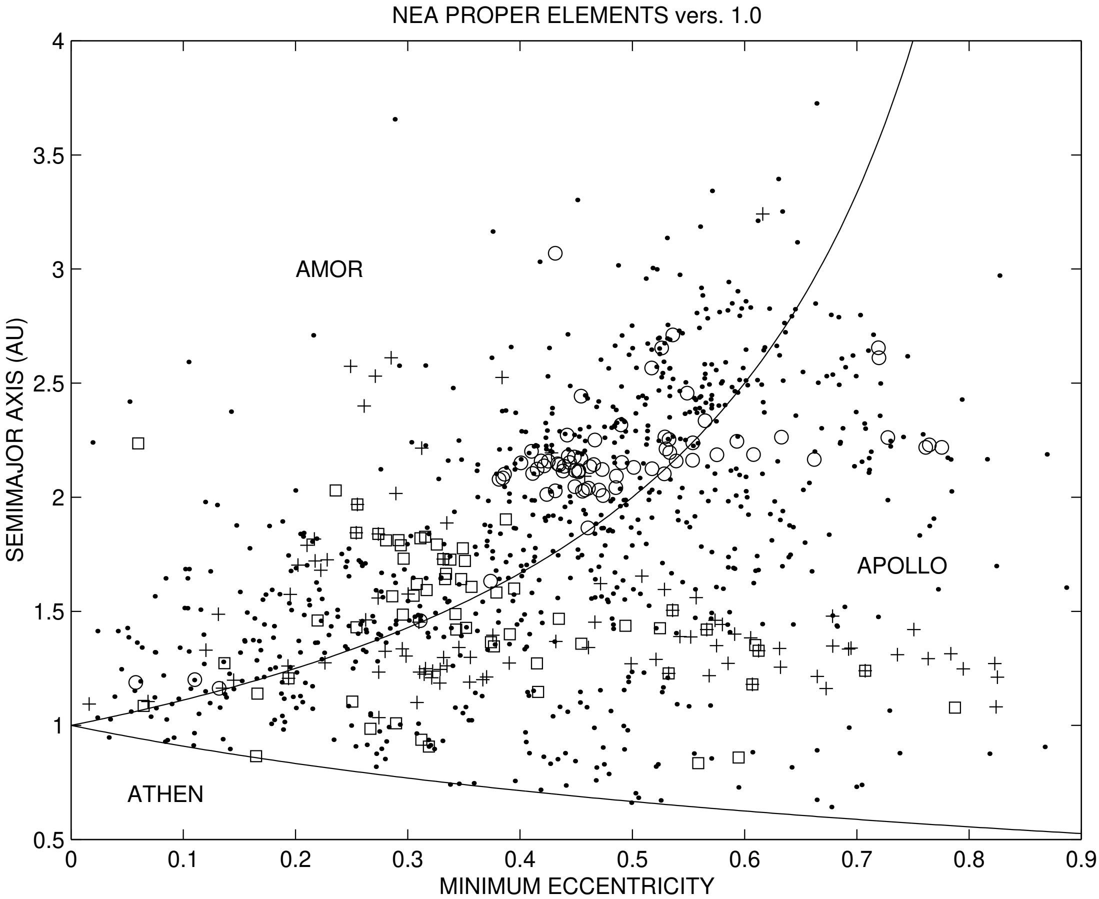

Figure 10: Map of the Near Earth Asteroids affected by one or more of the main secular resonances in the phase space (minimum e, a): the asteroids with $g \simeq g_6$ have been plotted with a circle, the ones with $s \simeq s_6$ with a square and the ones with $g \simeq g_5$ with a cross.